Notes on Vector Integrals

The two types of vector integrals that we will perform in the physics 130 series are **path** and **surface** integrals. Both types of integrals give scalar results. Here we summarize the main ideas for each.

Path Integrals

To calculate a path integral one needs to two "things": a path, and a vector defined at each point on the path. To carry out a path integral, one first divides up the path into N small "straight" segments. We label each segment with the integer index i. At the location of each segment, a vector is defined. We label the vector at the location of the "i"th segment as $\vec{v}_i$. Each small "straight" segment can be represented by a vector, which we label as $\vec{\Delta} l_i$.

The path integral is the limit as $N \to \infty$ (and $|\vec{\Delta} l_i| \to 0$) of the sum of $\vec{v}_i \cdot \vec{\Delta} l_i$ over all the segments. The path integral is written as $\int \vec{v} \cdot \vec{dl}$ or as $\int \vec{v} \cdot \vec{dr}$. Stated mathematically:

$$\int_{\vec{a}}^{\vec{b}} \vec{v} \cdot \vec{dl} \equiv \lim_{N \to \infty} \sum_{i=1}^{N} \vec{v}_i \cdot \vec{\Delta} l_i \tag{1}$$

Where the path starts at the location $\vec{a}$ and ends at the location $\vec{b}$. In our first year physics classes we will only evaluate simple path integrals. For example:

Straight line path and a constant vector field

Suppose there is a region of space where the vector field is constant. That is, $\vec{v}$ is the same at every location in the space. Let $\vec{l}$ be the vector representation the straight path line. That is, $\vec{l} = \vec{b} - \vec{a}$. In this case:

$$\int_{\vec{a}}^{\vec{b}} \vec{v} \cdot \vec{dl} = \vec{v} \cdot \vec{l} \tag{2}$$

If the angle between $\vec{v}$ and $\vec{l}$ is θ, then the path integral is simply $|\vec{v}||\vec{l}|cos\theta$.

A path that is perpendicular to the vector field

In some cases we will choose a path that is always perpendicular to the vector field all along the entire path. In this case, $\vec{v}_i \cdot \vec{\Delta} l_i$ is zero for each segment i. Thus the path integral is zero.

A path that follows the vector field

In some cases we will choose a path such that the vector $\vec{v}_i$ is parallel (or tangent) to the path. In this case the dot product is one, since $cos(0) = 1$. The path integral reduces to just an integral of the magnitude along the path. In the special case in which the magnitude is constant along the path, then $\int \vec{v} \cdot \vec{dl} = |\vec{v}|(path\ length)$.

Some of the path integrals that we will encounter in our Phy130 series are as follows:

1. The work done by a force along a path. Since work is the component of the force in the direction of the path, the path integral is: $\int \vec{F} \cdot \vec{dl}$.

2. The electostatic potential energy difference between two points. The vector field is the electric field $\vec{E}$, and the path integral is: $\int \vec{E} \cdot \vec{dl}$. Since the electric field is just the force per charge, the integral is the potential energy difference per charge, which is the voltage difference.

3. The magnetic field around a closed path. If the path returns to its starting point, it is called a closed path. In this case, the integral is given the symbol $\oint \vec{v} \cdot \vec{dl}$. Ampere's law involves taking a closed path integral of the magnetic field. $\oint \vec{B} \cdot \vec{dl}$. In our applications of Ampere's law, we will choose paths that follow the magnetic field. Often the magnetude of the magnetic field will not change along the path, and the path integral will just be $|\vec{B}|$ times the path length.

Surface Integrals

An important mathematical quantity for the formulation of the laws of electrodynamics is the surface integral. The surface integral is often called flux. In order to define a surface integral, one needs two things:

1. *A vector field*: A vector field is a "field of vectors". That is, at every point in space there is a vector defined. We will encounter two vector fields in Phy133, the electric field $\vec{E}$ and the magnetic field $\vec{B}$. Other vector fields in physics include the velocity vector in a fluid and the graviational field to name a few.

2. *A surface.* The surface can be a flat surface, a curved surface, or a closed surface.

We define a surface integral to be a scalar quantity. The vector field is a vector. So to make a scalar out of a vector, we will need to use the scalar (or dot) product with another vector. We will do this by associating a vector with the surface. This will be possible if the surface is flat, or very small such that it can be considered flat. Consider a flat surface. **We define the area vector associated with the flat surface as a vector whose direction is perpendicular to the surface and whose magnitude equals the area of the surface**.

We will first define the surface integral for a constant vector field and a flat surface. Suppose there exists a constant vector field $\vec{V}$. This means that at every point in space there is a vector $\vec{V}$, the same vector at all points in space. Also suppose there is a flat surface that has an area A. The area vector $\vec{A}$ associated with this surface is perpendicular to the surface. Surface integral, or flux, is often labeled as Φ, and is defined as:

$$\Phi \equiv \vec{V} \cdot \vec{A} \tag{3}$$

Remember this definition holds for a constant vector field and a flat surface. Φ is a scalar. The dot product is maximized when the two vector fields are parallel to each other. Thus, the flux will be maximized when the vector field is perpendicular to the surface. The dot product is zero when the two vector fields are perpendicular to each other. Thus, Φ will be zero when the surface is "parallel" to the vector field. Qualitatively, one can think of the flux as how much of the vector field passes through the surface. If we let θ be the angle between the area vector and the vector field,

$$\Phi \equiv |\vec{V}||\vec{A}|cos(\theta) \tag{4}$$

for a constant vector field and a flat surface. How do we generalize this definition to include a vector field that can vary from point to point in space and to a surface that is not necessarily flat? We can do this by dividing up the surface S into very small surfaces, ΔS_i where i is the i'th little surface piece. If the ΔS_i are small enough, they will be very close to being a flat surface. The smaller they are, the closer they will be to being flat. Also, if the surface is small enough, the vector field will be approximately constant on it. Thus, if $\vec{V_i}$ is the vector at the center of the i'th surface, we can define the flux of the vector field $\vec{V}(\vec{r})$ over the surface S as

$$\Phi_V = \lim_{\Delta S_i \to 0} \sum_i \vec{V_i} \cdot \Delta \vec{A_i} \tag{5}$$

where $\Delta\vec{A}_i$ is the area vector associated with the i'th surface. Thus, the limiting process allows us to extend our definition of flux for a flat surface to a curved one. As the small surfaces ΔS_i get smaller and smaller, the surfaces approach flat ones and the vector field approaches a constant one over the surface. The number of surfaces goes to infinity and the sum becomes an integral. It is this integral that is called a **surface integral** and is written as

$$\begin{aligned} \Phi_V &= \lim_{\Delta S_i \to 0} \sum_i \vec{V}_i \cdot \Delta\vec{A}_i \\ &= \int\!\!\int_S \vec{V}(\vec{r}) \cdot d\vec{A} \end{aligned}$$

The two integral signs denote that it is a two dimensional integral over the surface S. Often $\vec{V}(\vec{r})$ is written simply as $\vec{V}$. However, one must remember that $\vec{V}$ is a vector field and in general can depend on its location on the surface. The subscript V on Φ reminds us that the surface integral is for the vector field $\vec{V}$. Remember, when discussing flux, one needs to specify the **vector field** and the **surface**. The above integral can be complicated. In this course, we will only consider surface integrals that are easy to compute. We will usually choose our surfaces such that the vector field is perpendicular to the surface, and has a simple dependence over the surface.

When the surface is closed, we put a circle around the double integral sign: $\oint\oint \vec{v} \cdot d\vec{A}$. Gauss's law uses a closed surface integral to express the laws of electrostatics.

The surface integral is very important in Phy133. One can express Coulomb's law in terms of a closed surface integral (Gauss's Law). If the electric and/or magnetic fields change in time, then important "physics" occurs. The easiest way (in Phy133) to describe this physics is to use the time rate of change of a surface integral: Faraday's law and Maxwell's displacement current.