Notes on Vector Algebra

In physics, one often introduces vectors as quantities that have direction and magnitude. This is a limited concept, and will eventually need to be generalized. We will start with this simple application of vectors (vector spaces) and generalize when we need to. The first generalization we need to point out is that in order for a quantity to be a vector it must have certain properties of addition and multiplication by a scalar. We will show during our discussions that certain physical quantites have the properties required of a vector. Some examples are: displacement, velocity, acceleration, force, the electric and magnetic fields, to name a few.

We donate a vector quantity by an arrow over it: $\vec{A}$ represents the vector "A". The magnitude of the vector $\vec{A}$ is written as: $|\vec{A}|$. We can describe the direction using angles with respect to a reference direction. As described below, it is often convenient to describe a vector using unit vectors and "components".

Multiplication of a Vector with a scalar

Let c be a scalar and $\vec{A}$ be a vector. A scalar times a vector is a vector: $c\vec{A}$ is a vector. The magnitude of $c\vec{A}$ equals the magnitude of c times the magnitude of $\vec{A}$: $|c\vec{A}| = |c||\vec{A}|$. The direction of $c\vec{A}$ is the same as $\vec{A}$ **if** $c > 0$. If $c < 0$ then the direction of $c\vec{A}$ is opposite to the direction of $\vec{A}$.

Vector Addition

If two vectors are added together, the result is another vector. Vector addition must also have the property that is it commutative. That is, if $\vec{A}$ and $\vec{B}$ represent two vectors, then $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. One way to add vectors that satisfies the commutative property is the "tail-to-tip" method that will be described in lecture that works for vectors with dimension less than or equal to 3. If a vector has a dimension ≤ 3 it can be represented by an arrow on a piece of paper. Let $\vec{C} = \vec{A} + \vec{B}$. Then $\vec{C}$ is found by placing the tail of $\vec{B}$ at the tip of $\vec{A}$. The arrow that has its tail at the tail of $\vec{a}$ and its tip at the tip of $\vec{B}$ is the vector sum $\vec{C}$.

Although the "tail-to-tip" method is a way to add vectors, it is often inconvenient and difficult to use if more than two vectors are being added. An easy way to add vectors is to define unit vectors or basis vectors. A unit vector is a vector of magnitude 1, and is represented by a "hat" over the variable. We usually define the unit vector $\hat{i}$ as a vector of length 1 that points in the direction of the x-axis. The unit vector $\hat{j}$ is defined to point along the y-axis with magnitude 1, and $\hat{k}$ as a unit vector which points in the direction of the z-axis. Any vector in three dimensions can be written in terms of these unit (or basis) vectors. If (A_x, A_y, A_z) are the coordinates of the "tip" of the vector $\vec{A}$, then $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$.

To add vectors, one just adds their components. We demonstrate with an example of vector addition. Let vector $\vec{A}$ have a magnitude of 150 units at an angle of 30 degrees N of E, vector $\vec{B}$ have a magnitude of 100 units at an angle of 45 degrees N of W, and vector $\vec{C}$ have a magnitude of 200 units at an angle of 80 degrees S of E. Find the vector $\vec{D} = \vec{A} + \vec{B} + \vec{C}$, which is the sum of the three vectors. First we express the vectors in terms of the unit vectors:

$$\begin{aligned}\vec{A} &= 150cos(30)\hat{\imath} + 150sin(30)\hat{\jmath} \\ &= 130\hat{\imath} + 75\hat{\jmath}\end{aligned}$$

Similarly,

$$\begin{aligned}\vec{B} &= -100cos(45)\hat{\imath} + 100sin(45)\hat{\jmath} \\ &= -70.7\hat{\imath} + 70.7\hat{\jmath}\end{aligned}$$

and

$$\begin{aligned}\vec{C} &= 200cos(80)\hat{\imath} - 200sin(80)\hat{\jmath} \\ &= 34.7\hat{\imath} - 197\hat{\jmath}\end{aligned}$$

To add the three vectors, one simple adds the components:

$$\begin{aligned}\vec{A} + \vec{B} + \vec{C} &= (130 - 70.7 + 34.7)\hat{\imath} + (75 + 70.7 - 197)\hat{\jmath} \\ \vec{D} &= 94\hat{\imath} - 51.3\hat{\jmath}\end{aligned}$$

The sum can be expressed in terms of the unit vectors, or as a magnitude and direction: $|\vec{D}| = \sqrt{94^2 + 51.3^2} = 107$ units. The vector points in the fourth quadrant, with the angle $tan^{-1}\theta = 51.3/94$, or $\theta = 28.6°$. Since $\vec{D}$ is in the fourth quadrant, $\theta = 28.6°$ S of E.

Expressing the vectors using unit (or basis) vectors is probably the most convenient way to perform operations with them. In general, if $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$, and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$, then the sum $\vec{A} + \vec{B}$ equals

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} \tag{1}$$

for subtraction, replace the "+" with a "-". Using unit (or basis) vectors is particulary useful when adding (or subtraction) more than two vectors.

It is important to remember that **for a complete description of a vector in two dimensions, two numbers are needed**: a magnitude plus direction, two components, etc. For example, the vector $\vec{D}$ above can be expressed as: $94\hat{\imath} - 51.3\hat{\jmath}$, or 107 units at a direction 28.6° S of E, or 107 units at an angle of 331.4° clockwise from the x-axis.

It should be pointed out that all quantities with a magnitude and direction are not necessarily vectors. To be a vector, a quantity needs to have certain propeties. Also, here we have limited our applications to two and three dimensional vectors. These ideas can be generalized to any number of dimensions (including infinity). For higher dimensional vector spaces, it is not always useful to think of the direction of a vector.

We end this section with an example of a quantity that has direction and magnitude, but does not have the properties required of a vector. The addition property of vectors must be commutative: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. Rotations about an axis, which have a magnitude and direction, do not have this property. Let the arrow representing a rotation be defined in the following way: the direction is in the direction of the axis of rotation and the magnitude is equal to the amount of rotation about the axis (measured counter-clockwise). Let A be a rotation of $\pi/2$ counter-clockwise about the x-axis. The "vector" representing this rotation is $\vec{A} = (\pi/2)\hat{\imath}$. Let B be a rotation of $\pi/2$ counter-clockwise about the y-axis. The "vector" representing this rotation is $\vec{B} = (\pi/2)\hat{\jmath}$. If vector addition is defined as successive rotations, $\vec{A} + \vec{B} \neq \vec{B} + \vec{A}$. That is if rotation A is first applied to an object then rotation B, the orientation of the object is different than if rotation B is first applied then rotation A. Try it out with your physics book.

There are a number of quantities in classical physics that behave as vectors: displacement, velocity, acceleration, force, the electric field and the magnetic field to name a few. You will encounter these during your first year of physics. Although they represent different physical quantites, they all add like the method described here. Vector addition pertains to many different physical quantites, it is a "universal mathematics", and its importance cannot be understated.

Vector Scalar Product

The vector scalar product is an operation between two vectors that produces a scalar. Let $\vec{A}$ and $\vec{B}$ be two vectors. We denote the scalar product as $\vec{A} \cdot \vec{B}$. There are a number of ways to derive a scalar quantity from two vectors. One can use $|\vec{A}|$ and $|\vec{B}|$. However, if we define the scalar product as the product of the magnitudes, then the angle does not play a role. If we call θ the angle between the vectors, then we could use cos(θ) or sin(θ) in our definition. If we want $\vec{A} \cdot \vec{B}$ to equal $\vec{B} \cdot \vec{A}$ then we need a trig function that is symmetric in θ. Since cos(-θ) equals cos(θ) it is the best choice. The scalar product is defined as

$$\vec{A} \cdot \vec{B} \equiv |\vec{A}||\vec{B}|cos(\theta) \tag{2}$$

where θ is the angle between the two vectors.

The scalar product can be negative, positive, or zero. It is the magnitude of $\vec{A}$ times the component of $\vec{B}$ in the direction of $\vec{A}$. If the vectors are perpendicular, then the scalar product is zero. The scalar products between the unit vectors are:

$$\hat{\imath} \cdot \hat{\jmath} = \hat{\imath} \cdot \hat{k} = \hat{\jmath} \cdot \hat{k} = 0 \tag{3}$$

and

$$\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1 \tag{4}$$

If the vectors are expressed in terms of the unit vectors (i.e. their components: $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$, the distributive property of the scalar product gives

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z \tag{5}$$

Vector Cross Product

The cross product of vector $\vec{A}$ and $\vec{B}$ is a vector. The magnitude of $\vec{A} \times \vec{B}$ equals the magnitude of $\vec{A}$, $|\vec{A}|$ times the magnitude of $\vec{B}$, $|\vec{B}|$, times the sin of the angle between them:

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|sin(\theta) \tag{6}$$

where the angle θ is the angle between vector $\vec{A}$ and vector $\vec{B}$. The angle goes from $\vec{A}$ to the vector $\vec{B}$. The direction of $\vec{A} \times \vec{B}$ is determined by the "right hand" rule described above. Note that since $sin(-\theta) = -sin(\theta)$,

$$\vec{B} \times \vec{A} = -\vec{A} \times \vec{B} \tag{7}$$

Note that the cross product is maximized when $\vec{A}$ is perpendicular to $\vec{B}$, and zero when θ is zero or 180°. The unit vectors have simple cross product properties:

$$\begin{aligned}
\hat{\imath} \times \hat{\jmath} &= \hat{k} \\
\hat{\jmath} \times \hat{k} &= \hat{\imath} \\
\hat{k} \times \hat{\imath} &= \hat{\jmath} \\
\hat{\imath} \times \hat{\imath} &= 0 \\
\hat{\jmath} \times \hat{\jmath} &= 0 \\
\hat{k} \times \hat{k} &= 0
\end{aligned}$$

We can express $\vec{A}$ and $\vec{B}$ in terms of the unit vectors: $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$. If we use the above forumlas for the cross products of the unit vectors, after "foil"ing we have:

$$\begin{aligned}
\vec{A} \times \vec{B} &= (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} \\
&\quad +(A_xB_y - A_yB_x)\hat{k}
\end{aligned}$$

This formula can be succinctly written in terms of the determinant of a matrix:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{8}$$

Comments and Summary

We have covered all the vector algebra operations that we will need for our first year physics series. Let's summarize and comment on them:

Vector Addition: When two vectors are added, the result is another vector. Vector addition is commutative, $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. The "tail-to-tip" method satisfies the commutative property. In terms of components, $(\vec{A} + \vec{B})_x = A_x + B_x$, $(\vec{A} + \vec{B})_y = A_y + B_y$, and $(\vec{A} + \vec{B})_z = A_z + B_z$. If physical quantites are vectors, addition of vectors is used to find the NET sum of the quantites (e.g. the net force).

Multiplication of a scalar and a Vector: Multiplication of a vector by a scalar c changes the magnitude of the vector by the factor $|c|$. If $c > 0$, then the direction remains the same, but if $c < 0$ the direction is reversed. Multiplication by a scalar enables one to define unit vectors, whose linear combinations span the vector space.

Scalar product of two vectors: The scalar product of one vector with another vector yields a scalar. Since two vectors are involved, the scalar product must depend on the magnitudes of each vector and the relative direction of one to the other. If θ is the angle between the vectors, the scalar product could depend on either $sin(\theta)$ or $cos(\theta)$. Since the scalar product should be commutative, $cos(\theta)$ is the only choice since $cos(-\theta) = cos(\theta)$. Thus in order to be commutative, the scalar product must be $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|cos(\theta)$. In terms of components, $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$. In physics, the scalar product represents the component of one vector in the direction of another (times its magnitude). It allowed us to express the work done by a force in acting in a certain direction easily ($\vec{F} \cdot \Delta\vec{r}$).

Vector product of two vectors: The vector product of one vector with another yields a vector. Since two vectors are involved here also, the vector product must depend on the magnitudes of each vector and the relative direction of one to the other. Since the result must be a vector, there is only one special direction: the direction perpendicular to $\vec{A}$ and $\vec{B}$. Should $sin(\theta)$ or $cos(\theta)$ be used? Theta equal to zero helps us determine the proper choice. If $\vec{A}$ is parallel to $\vec{B}$ then there is no special direction, so the vector product must be zero in this case. The trig function that is zero for $\theta = 0$ is $sin(\theta)$. Thus, the only way to define a vector product of two vectors is to use the sin function as described above.

These vector operations will be very important in describing the electro- magnetic interaction. This is because the magnetic part of the interaction depends on the velocity of the particles. With so many vectors involved, $\vec{F}$, $\vec{r}$, and $\vec{v}$, the scalar and vector products are important. It will also be necessary to consider derivatives of vectors, i.e. vector calculus.