

## Notes on LRC circuits and Maxwell's Equations

Last quarter, in Phy133, we covered electricity and magnetism. There was not enough time to finish these topics, and this quarter we start where we left off and complete the classical treatment of the electro-magnetic interaction. We begin with a discussion of circuits which contain a capacitor, resistor, and a significant amount of self-induction. Then we will revisit the equations for the electric and magnetic fields and add the final piece, due to Maxwell. As we will see, the missing term added by Maxwell will unify electromagnetism and light. Besides unifying different phenomena and our understanding of physics, Maxwell's term lead the way to the development of wireless communication, and revolutionized our world.

### LRC Circuits

Last quarter we covered circuits that contained batteries and resistors. We also considered circuits with a capacitor plus resistor as well as resistive circuits that has a large amount of self-inductance. The self-inductance was dominated by a coiled element, i.e. an inductor. Now we will treat circuits that have all three properties, capacitance, resistance and self-inductance.

We will use the same "physics" we discussed last quarter pertaining to circuits. There are only two basic principles needed to analyze circuits.

**1. The sum of the currents going into a junction (of wires) equals the sum of the currents leaving that junction.** Another way is to say that the charge flowing into equals the charge flowing out of any junction. This is essentially a statement that charge is conserved.

1) The sum of the currents into a junction equals the sum of the currents flowing out of the junction.

The next principle involves the sum of the voltage drops around a closed wire loop. If the current is not changing, the sum is be zero. However, this is only correct if the current is not changing in the circuit. If the current is changing, then path integral of  $\oint \vec{E} \cdot \vec{dl}$  is not zero, but rather  $-d\Phi_m/dt$ . Thus, the Law involving voltage changes around a closed loop is:

$$\sum(\text{Voltage drops}) = - \oint \vec{E} \cdot \vec{dl} = \frac{d\Phi_m}{dt} \quad (1)$$

where  $\Phi_m$  is the magnetic flux through the closed path of the loop. The direction of adding the voltage drops determines the direction of positive magnetic flux. Last quarter we applied these two laws of physics to circuits containing resistors and capacitors, as well as a circuit containing a resistor and inductor. You should review what was covered. Now we consider a circuit that has a capacitor and an inductor.

### *L-C Resonance Circuit*

An important application is a circuit that has a large self-inductance and a capacitor. We start with the simplest case. For the circuit to have a large self-inductance, we add a coil (or solenoid) that has an inductance  $L$ . The circuit contains only the capacitor connected to the solenoid. We assume that the self-inductance of the rest of the circuit is negligible compared to the solenoid, so that the net self-inductance of the circuit is  $L$ . If the self-inductance of the circuit is  $L$ , then we have  $\Phi_m = LI$ . The sum of the the voltage changes around the loop in the direction of "+" current becomes

$$\sum(\text{Voltage drops}) = +L\frac{dI}{dt} \quad (2)$$

where  $I$  is the current in the circuit. Note that the sign on the right side of the equation is "+" if the direction of voltage changes is in the same direction as the current. It would be minus if the other direction were chosen. Let the capacitor have a capacitance of  $C$ . Let  $\pm Q$  be the charge on the plates of the capacitor. Applying the voltage equation to the voltage changes around the circuit gives:

$$V_c = L\frac{dI}{dt} \quad (3)$$

where  $V_c$  is the voltage across the capacitor. To solve this equation, we need to have another relationship between  $V_c$  and  $I$ . Expressing  $V_c$  and  $I$  in terms of the charge on the capacitor plates will give us the connection we need. If  $\pm Q$  are the charges on each plate of the capacitor, we have  $V_c = Q/C$ .

The current is the rate of change of the charge on the capacitor. If we take  $+I$  in a direction away from the positive plate of the capacitor, then  $I = -(dQ)/(dt)$ . Substituting into the equation above gives:

$$V_c = L\frac{dI}{dt}$$

$$\begin{aligned}\frac{Q}{C} &= -L \frac{d^2 Q}{dt^2} \\ \frac{d^2 Q}{dt^2} &= -\frac{Q}{LC}\end{aligned}$$

The above equation is a relatively simple differential equation. The solution  $Q(t)$  is a function whose second derivative is minus itself. Functions with this property are sinusoidal functions. How do we include the factor  $1/(LC)$ . We can guess the solution by remembering how the chain rule works. You should verify that the function:  $Q(t) = A \sin(t/\sqrt{LC})$  as well as the function  $Q(t) = B \cos(t/\sqrt{LC})$  are solutions to the equation above. Any linear combination of these two solutions is also a solution. So the most general solution is

$$Q(t) = A \sin\left(\frac{t}{\sqrt{LC}}\right) + B \cos\left(\frac{t}{\sqrt{LC}}\right) \quad (4)$$

where  $A$  and  $B$  are constants that depend on the initial conditions. Since the equation is a second order differential equation, there will be two "integration constants". If the initial conditions are such that the initial charge on the capacitor is  $Q_0$  and that the current  $I$  is initially zero, then  $B = Q_0$  and  $A = 0$ . For these initial conditions the solution is

$$Q(t) = Q_0 \cos\left(\frac{t}{\sqrt{LC}}\right) \quad (5)$$

From a knowledge of  $Q(t)$ , the voltage across the capacitor and the current in the circuit can be determined. The voltage across the capacitor is

$$V_c(t) = \frac{Q_0}{C} \cos\left(\frac{t}{\sqrt{LC}}\right) = V_0 \cos\left(\frac{t}{\sqrt{LC}}\right) \quad (6)$$

and the current in the circuit is

$$\begin{aligned}I(t) &= -\frac{dQ}{dt} \\ I(t) &= \frac{Q_0}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right)\end{aligned}$$

These are interesting results. The voltage across the capacitor and the current in the circuit oscillate back and forth. The period  $T$  of this oscillation is given by

$$\begin{aligned}\frac{T}{\sqrt{LC}} &= 2\pi \\ T &= 2\pi\sqrt{LC}\end{aligned}$$

The frequency of the oscillation is given by  $f = 1/T = 1/(2\pi\sqrt{LC})$ . This frequency is the resonance frequency of the circuit. Note that the quantity  $1/\sqrt{LC}$  occurs in the argument of the sin (or cos) function. It is convenient to define  $\omega_0 \equiv 1/\sqrt{LC}$ . With this definition, we have for  $I(t)$  and  $V_c(t)$ :

$$\begin{aligned}I(t) &= \frac{Q_0}{\sqrt{LC}}\sin(\omega_0 t) \\ V_c(t) &= V_0\cos(\omega_0 t)\end{aligned}$$

where

$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \tag{7}$$

A couple of things to note.

- 1) The quantity  $\omega_0$  has units of  $1/time$ . It enters in the argument of the sinusoidal functions in the same way as angular frequency would be for an object moving in a circle. Nothing is rotating here, and there are no physical angles. So when we refer to  $\omega$  as the angular frequency, just think of it as  $2\pi$  times the frequency.
- 2) Both the current in the circuit and  $V_c$  are sinusoidal functions, but they are out of phase by  $90^\circ$ . One varies as  $\sin(\omega t)$  and the other as  $\cos(\omega t)$ .
- 3) Initially  $I = 0$ , and all the energy is in the electric field of the capacitor, and the magnetic field in the solenoid is zero. Then at  $t = T/2$ , there is no charge on the capacitor, the current is maximized, and all the energy is in the magnetic field of the solenoid. The energy is being transferred back and forth between the capacitor (electric field energy) and the inductor (magnetic field energy).

In any real circuit there will be resistance. The energy of the  $LC$  circuit will be gradually dissipated by resistive elements. Next we consider what happens if we

include resistance in the circuit.

### *R-L-C Resonance Circuit*

Consider now a series circuit that has a capacitor, capacitance  $C$ , a resistor, resistance  $R$ , and a solenoid, self-inductance  $L$ . As before, we will assume that the self-inductance of the whole circuit is approximately that of the solenoid,  $L$ . Let's also neglect the resistance of the solenoid. Now, we can apply the physics of circuits to the  $R - L - C$ -series circuit.

$$\sum(\text{Voltage drops}) = L \frac{dI}{dt} \quad (8)$$

if the path for the voltage changes is in the same direction as the "+" direction of the current. As before, we take the "+" direction of the current  $I$  to be away from the positive side of the capacitor. Adding up the changes in voltage in the direction of "+" current we have:

$$V_c - RI = L \frac{dI}{dt} \quad (9)$$

$V_c$  is positive, since the path goes from the negative to the positive side of the capacitor (a gain in voltage). The voltage drop across the resistor is  $-RI$  because the path is in the direction of the current (i.e. a drop in voltage). As before, substituting  $V_c = Q/C$  and  $I = -(dQ/dt)$  gives

$$\begin{aligned} \frac{Q}{C} + R \frac{dQ}{dt} &= -L \frac{d^2Q}{dt^2} \\ L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} &= 0 \\ \frac{d^2Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{Q}{LC} &= 0 \\ \frac{d^2Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \omega_0^2 Q &= 0 \end{aligned}$$

where  $\omega_0 \equiv 1/\sqrt{LC}$ .

We can guess what the solution to the differential equation above should be. With  $R = 0$ , the solution is sinusoidal. If  $L = 0$ , the solution is a decaying exponential. So

we can guess that the solution might be a sinusoidal function multiplied by a decaying exponential. There are different ways to solve this differential equation. I will show two. The first way is rather complicated, but uses real quantities. The second way is simpler, and uses complex numbers. Complex numbers will be very useful in future calculations.

*Using real quantities*

Let's try to find a solution of the form  $Q(t) = e^{-\gamma t}(A\cos(\omega t) + B\sin(\omega t))$ , where all quantities are real. The constants  $A$  and  $B$  should depend on the initial conditions. That is, we need to find a solution that is valid for any  $A$  and  $B$ . The constants  $\gamma$  and  $\omega$  therefore need to exist such that  $Q(t)$  is a solution for any  $A$  and  $B$ . Differentiating our expression for  $Q(t)$  gives:

$$\begin{aligned} Q(t) &= e^{-\gamma t}(A\cos(\omega t) + B\sin(\omega t)) \\ \frac{dQ}{dt} &= e^{-\gamma t}((B\omega - A\gamma)\cos(\omega t) - (B\gamma + A\omega)\sin(\omega t)) \\ \frac{d^2Q}{dt^2} &= e^{-\gamma t}((A(\gamma^2 - \omega^2) - 2B\omega\gamma)\cos(\omega t) + (B(\gamma^2 - \omega^2) + 2A\omega\gamma)\sin(\omega t)) \end{aligned}$$

Now, substituting  $Q$ ,  $dQ/(dt)$ , and  $d^2Q/(dt^2)$  into the differential equation, gives

$$\begin{aligned} e^{-\gamma t}\cos(\omega t)((A(\gamma^2 - \omega^2 - R\gamma/L + \omega_0^2) + B\omega(R/L - 2\gamma) + \\ e^{-\gamma t}\sin(\omega t)(A\omega(2\gamma - R/L) + B(\gamma^2 - \omega^2 - R\gamma/L + \omega_0^2)) = 0 \end{aligned}$$

This equation must be true for all times  $t$ . Therefore the terms multiplying the  $\sin(\omega t)$  and the  $\cos(\omega t)$  must each be equal to zero:

$$\begin{aligned} A(\gamma^2 - \omega^2 - R\gamma/L + \omega_0^2) + B\omega(R/L - 2\gamma) &= 0 \\ A\omega(2\gamma - R/L) + B(\gamma^2 - \omega^2 - R\gamma/L + \omega_0^2) &= 0 \end{aligned}$$

The condition that these two equations to be valid for any  $A$  and  $B$  require that:

$$\begin{aligned} 2\gamma - R/L &= 0 \\ \gamma &= \frac{R}{2L} \end{aligned}$$

and

$$\begin{aligned}\gamma^2 - \omega^2 - R\gamma/L + \omega_0^2 &= 0 \\ \frac{R^2}{4L^2} - \omega^2 - \frac{R^2}{2L^2} + \omega_0^2 &= 0 \\ \omega^2 &= \omega_0^2 - \frac{R^2}{4L^2} \\ \omega &= \sqrt{\omega_0^2 - (R/(2L))^2}\end{aligned}$$

So the general solution for the *LRC* decaying circuit is

$$\begin{aligned}Q(t) &= e^{-R/(2L)t}(A\cos(\sqrt{\omega_0^2 - (R/(2L))^2}t) + B\sin(\sqrt{\omega_0^2 - (R/(2L))^2}t)) \\ Q(t) &= A_0e^{-\gamma t}\cos(\omega t + \alpha)\end{aligned}$$

where  $\gamma = R/(2L)$  and  $\omega = \sqrt{\omega_0^2 - \gamma^2}$ . For the last step, we have combined a sin plus cos into a cos plus an angle. In terms of  $A$  and  $B$ , the constant  $A_0 = \sqrt{A^2 + B^2}$  and  $\tan(\alpha) = -B/A$ :

$$\begin{aligned}A_0\cos(\omega t + \alpha) &= A_0\cos(\omega t)\cos(\alpha) - A_0\sin(\omega t)\sin(\alpha) \\ &= A_0\cos(\omega t)\left(\frac{A}{A_0}\right) - A_0\sin(\omega t)\left(\frac{-B}{A_0}\right) \\ &= A\cos(\omega t) + B\sin(\omega t)\end{aligned}$$

$A$  and  $B$  are two legs of a right triangle. The hypotenuse is equal to  $\sqrt{A^2 + B^2} = A_0$ . The  $\cos(\alpha) = A/A_0$  and  $\sin(\alpha) = -B/A_0$ .

The math using real quantities is somewhat complicated. Now let's do the same thing with complex numbers.

*Using complex numbers*

The mathematical relationship that we use is "Euler's Equation":

$$e^{ix} = \cos(x) + i\sin(x) \tag{10}$$

where  $x$  is real and  $i = \sqrt{-1}$ . Guided by the reasoning we used with real numbers, we look for a solution of the form  $Q(t) = \text{Re}(ze^{\beta t})$ , where  $z$  and  $\beta$  are constants that

are complex numbers. Note, the solution  $Q(t)$  is real since we are taking the real part (Re) of a complex number. First, I'll show that the form we have chosen results in the same form as with the real numbers. Since  $z$  is complex, we can write it as  $z = A_0 e^{i\alpha}$ , where  $A_0$  and  $\alpha$  are real. Since  $\beta$  is complex, we can write it as  $\beta = -\gamma + i\omega$ , where  $\gamma$  and  $\omega$  are real. So, we have

$$\begin{aligned} Q(t) &= \operatorname{Re}(A_0 e^{i\alpha} e^{(-\gamma+i\omega)t}) \\ &= \operatorname{Re}(A_0 e^{-\gamma t} e^{i(\omega t+\alpha)}) \\ &= A_0 e^{-\gamma t} \operatorname{Re}(e^{i(\omega t+\alpha)}) \\ &= A_0 e^{-\gamma t} \cos(\omega t + \alpha) \end{aligned}$$

which is of the same form as that using the real parameters. That is,  $\operatorname{Re}(\beta) \rightarrow -\gamma$ ,  $\operatorname{Im}(\beta) \rightarrow \omega$ .  $z$  (i.e.  $A_0$  and  $\alpha$ ) is a complex constant that depends on the initial conditions as before. We can now substitute the complex function  $ze^{\beta t}$  into the differential equation and find a solution. Since the real and imaginary parts are independent, they will each satisfy the differential equation. Both the real and imaginary parts will be solutions for  $Q(t)$ , and we will take the real part. Substituting  $ze^{\beta t}$  into our differential equation:

$$\begin{aligned} z\left(\frac{d^2 e^{\beta t}}{dt^2} + \frac{R}{L} \frac{de^{\beta t}}{dt} + \omega_0^2 e^{\beta t}\right) &= 0 \\ \frac{d^2 e^{\beta t}}{dt^2} + \frac{R}{L} \frac{de^{\beta t}}{dt} + \omega_0^2 e^{\beta t} &= 0 \end{aligned}$$

The constant  $z$  factors out of each term since it is a constant. Differentiating the exponential functions is easy:

$$\begin{aligned} e^{\beta t}\left(\beta^2 + \frac{R}{L}\beta + \omega_0^2\right) &= 0 \\ \beta^2 + \frac{R}{L}\beta + \omega_0^2 &= 0 \end{aligned}$$

and we are left with a quadratic equation for  $\beta$ , whose solution is

$$\begin{aligned} \beta &= \frac{-R/L \pm \sqrt{(R/L)^2 - 4\omega_0^2}}{2} \\ &= -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \omega_0^2} \end{aligned}$$



If the resistance is small such that  $R/(2L) < \omega_0$ , then the argument in the square root is negative. Defining  $\gamma = R/(2L)$  and  $\omega = \sqrt{\omega_0^2 - \gamma^2}$  we have

$$\beta = -\gamma \pm i\omega \quad (11)$$

We take the real part of  $ze^{\beta t}$  for  $Q(t)$ :

$$\begin{aligned} Q(t) &= \operatorname{Re}(ze^{(-\gamma \pm i\omega)t}) \\ &= \operatorname{Re}(A_0 e^{i\alpha} e^{(-\gamma \pm i\omega)t}) \\ &= \operatorname{Re}(A_0 e^{-\gamma t} e^{\pm i(\omega t + \alpha)}) \\ &= A_0 e^{-\gamma t} \operatorname{Re}(e^{\pm i(\omega t + \alpha)}) \\ &= A_0 e^{-\gamma t} \cos(\omega t + \alpha) \end{aligned}$$

which is the same result we obtained using only real quantities. Using complex numbers greatly simplifies the solution. We will use complex numbers a few more times this quarter to solve differential equations and to add sinusoidal functions having the same frequency.

A final point mention is that only  $\gamma$  and  $\omega$  depend on the circuit elements  $R$ ,  $L$ , and  $C$ .  $A_0$  and  $\alpha$  depend on the initial conditions.

Next we will consider an  $R, L, C$  circuit that has a sinusoidal voltage source.

#### *RL circuit with sinusoidal voltage source*

As a final example of  $LRC$  series circuits, we will add a sinusoidal voltage source to the circuit. Let's first start with a circuit that has a self-inductance  $L$  and a resistance  $R$ . The self-inductance is dominated by a solenoid as before. Let the frequency of the sinusoidal voltage source be  $f_d = \omega_d/(2\pi)$ , and the maximum voltage be  $V_m$ .

The same physics is true when there is a source of voltage present: 1) the current is the same through each circuit element since they are connected in series, and 2) the sum of the voltage changes around the loop equals the change of magnetic flux. Since the voltage source is sinusoidal, after any transient oscillations have damped out, the current in *the circuit will also be sinusoidal with the same frequency as the voltage source*. That is, the current through every element in the circuit can be written as

$$I(t) = I_m \cos(\omega_d t) \quad (12)$$

Note that  $I_m$  is the maximum amplitude that the current will have. We could have also chosen  $\sin(\omega_d t)$  and the end results would be the same. What we need to determine is the relationship between  $V_m$  and  $I_m$ , and the relative phase between the sinusoidal current and the sinusoidal voltages. For this, we use the voltage sum law. Equating the voltage changes around the loop to the change in magnetic flux through the loop gives:

$$\begin{aligned} V_d - IR &= L \frac{dI}{dt} \\ V_d &= IR + L \frac{dI}{dt} \end{aligned}$$

where the current  $I(t)$  is the same through the resistor and the solenoid. If  $I(t) = I_m \cos(\omega_d t)$ , then the voltage source satisfies

$$\begin{aligned} V_d &= IR + L \frac{dI}{dt} \\ V_d &= I_m R \cos(\omega_d t) - L \omega_d I_m \sin(\omega_d t) \\ V_d &= I_m (R \cos(\omega_d t) - L \omega_d \sin(\omega_d t)) \end{aligned}$$

We can simplify this expression by adding the sin and cos functions. A nice property of sinusoidal functions having the same frequency is that when they are added together the sum is a single sinusoidal function. We demonstrate this property for our circuit by using the trig identity  $\cos(\omega_d t + \phi) = \cos(\omega_d t) \cos(\phi) - \sin(\omega_d t) \sin(\phi)$ .

$$\begin{aligned} V_d &= I_m (R \cos(\omega_d t) - L \omega_d \sin(\omega_d t)) \\ V_d &= I_m \sqrt{R^2 + (L \omega_d)^2} \left( \frac{R}{\sqrt{R^2 + (L \omega_d)^2}} \cos(\omega_d t) - \frac{L \omega_d}{\sqrt{R^2 + (L \omega_d)^2}} \sin(\omega_d t) \right) \\ V_d &= I_m \sqrt{R^2 + (L \omega_d)^2} (\cos(\phi) \cos(\omega_d t) - \sin(\phi) \sin(\omega_d t)) \\ V_d &= I_m \sqrt{R^2 + (L \omega_d)^2} \cos(\omega_d t + \phi) \end{aligned}$$

The third line follows by noting that  $R$  and  $L \omega_d$  are two legs of a right triangle. The hypotenuse of this right triangle is  $\sqrt{R^2 + (L \omega_d)^2}$ , and  $\phi$  is an angle in the triangle. Since the cos function varies between  $\pm 1$ , the amplitude of the voltage source is

$V_m = I_m \sqrt{R^2 + (L\omega_d)^2}$ , and the voltage of the source *leads* the current by a phase  $\phi$ , where  $\tan(\phi) = L\omega_d/R$ .

Now let's add a capacitor in series with the resistor and solenoid. We need to determine what the voltage  $V_c$  across the capacitor is when a sinusoidal current  $I_m \cos(\omega_d t)$  flows through it. We know  $V_c = Q/C$  and  $I = +dQ/(dt)$ . Here there is a + sign in front of  $dQ/(dt)$  since current is flowing into the capacitor. Solving for  $Q$ :

$$\begin{aligned}\frac{dQ}{dt} &= I_m \cos(\omega_d t) \\ Q &= \int I_m \cos(\omega_d t) dt \\ Q &= \frac{I_m}{\omega_d} \sin(\omega_d t)\end{aligned}$$

The integrating constant will be the initial charge on the capacitor, which we take to be zero. Since  $V_c = Q/C$ , we have

$$V_c = \frac{I_m}{\omega_d C} \sin(\omega_d t) \quad (13)$$

Adding the voltage changes around the *RLC* series circuit loop gives

$$\begin{aligned}V_d - IR - V_c &= L \frac{dI}{dt} \\ V_d &= IR + L \frac{dI}{dt} + V_c\end{aligned}$$

The current through each element is  $I = I_m \cos(\omega_d t)$ , which yields for the voltages:

$$\begin{aligned}V_d &= I_m R \cos(\omega_d t) - L\omega_d I_m \sin(\omega_d t) + \frac{I_m}{\omega_d C} \sin(\omega_d t) \\ V_d &= I_m R \cos(\omega_d t) - I_m \left( L\omega_d - \frac{1}{\omega_d C} \right) \sin(\omega_d t)\end{aligned}$$

Finally, we can combine the sin and cos terms into one sinusoidal function as we did before with only the solenoid present.

$$V_d = I_m \sqrt{R^2 + \left( L\omega_d - \frac{1}{\omega_d C} \right)^2} \cos(\omega_d t + \phi) \quad (14)$$

where now

$$\tan(\phi) = \frac{L\omega_d - \frac{1}{\omega_d C}}{R} \quad (15)$$

Let's discuss our results.

1. The quantity  $\sqrt{R^2 + (L\omega_d - 1/(\omega_d C))^2}$  plays the role of resistance in our series AC sinusoidal circuit. This generalized resistance quantity is called the *impedance* of the circuit and usually given the symbol  $Z$ .

$$Z \equiv \sqrt{R^2 + (L\omega_d - \frac{1}{\omega_d C})^2} \quad (16)$$

Remember, however, that  $Z$  only has meaning for sinusoidal currents and voltage sources. If  $L = 0$  and in the absence of a capacitor,  $Z = R$ . Note that  $V_m = I_m Z$ .

2. The term  $L\omega_d$  is called the inductive reactance, and usually labeled as  $X_L \equiv L\omega_d$ . The inductive reactance has units of resistance (Ohms) and represents the effective inductive resistance of the solenoid for sinusoidal currents.
3. The term  $1/(\omega_d C)$  is called the capacitive reactance, and usually labeled as  $X_C \equiv 1/(\omega_d C)$ . The capacitive reactance has units of resistance (Ohms) and represents the effective capacitive resistance of the capacitor for sinusoidal currents.
4. With these definitions,  $Z = \sqrt{R^2 + (X_L - X_C)^2}$ , and  $\tan(\phi) = (X_L - X_C)/R$ .
5. For high driving frequencies,  $X_L = \omega_d L$  is the largest term. The circuit is mainly inductive. For low driving frequencies,  $X_C = 1/(\omega_d C)$  is the largest term, and the circuit is mainly capacitive.
6. If  $X_L > X_C$  (an inductive "L" situation) the relative phase  $\phi$  between the voltage and the current is positive: voltage leads current. If  $X_C > X_L$  (a capacitive "C" situation) the relative phase  $\phi$  between the voltage the current is negative: current leads voltage. Now you know about "ELI the ICE man".
7.  $Z$  becomes its smallest when  $X_L = X_C$ . In this case  $Z = R$ . The frequency for which this occurs is called the resonant frequency.  $X_L = X_C$ , i.e. the resonance condition, when the driving angular frequency is  $\omega_d = 1/\sqrt{LC}$ , or

$f_d = 1/(2\pi\sqrt{LC})$ . At this resonant frequency, the impedance takes on its smallest value and one gets the most current for the least voltage.

There is a nice geometric way to express the impedance using a "phasor diagram".  $R$  points along the "+x-axis",  $X_L$  points along the "+y-axis", and  $X_C$  points along the  $-y$ -axis.  $Z$  and  $\phi$  are found by adding the reactances and resistance like vectors.

One can do the analysis using complex numbers, if there is time I will cover this in class. For those interested,  $Z$  as well as  $V$ ,  $V_m$ ,  $I$  and  $I_m$  will be complex numbers. The current is  $I = I_m e^{i\omega_m t}$ , and the voltage is  $V = V_m e^{i\omega t}$ .  $R$  is real, but  $X_L = i\omega_d L$  and  $X_C = -i/(\omega_d C)$ . For a series connection, the complex impedance is  $Z = R + i(\omega_d L - 1/(\omega_d C))$ . The complex numbers  $V_m$ ,  $I_m$ , and  $Z$  are related by

$$\begin{aligned} V &= IZ \\ V_m e^{i\omega t} &= I_m e^{i\omega t} Z \\ V_m &= I_m Z \\ V_m &= I_m \sqrt{R^2 + (\omega_d L - \frac{1}{\omega_d C})^2} e^{i\phi} \end{aligned}$$

Taking the real parts of each side yields the same result we obtained using real numbers. For circuits that are combinations of series and parallel connections, using complex numbers makes the calculations much much easier.

#### *Power considerations for the series RLC circuit*

Energy will only be transferred into the resistive element in the circuit. Last quarter we derived that the power  $P$  transferred into a resistor with resistance  $R$  is  $P = I^2 R$ . A sinusoidal voltage produces a sinusoidal current, so the power varies in time as a sinusoidal squared function. If  $I = I_m \sin(\omega_m t)$ , then the power is

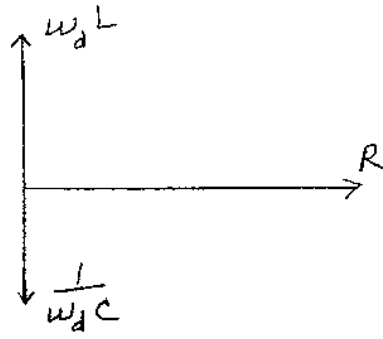
$$P = I_m^2 R \sin^2(\omega_m t) \quad (17)$$

and varies in time. We can calculate the average power,  $P_{ave}$  by averaging  $P$  over one period of oscillation:

$$P_{ave} = I_m^2 R \left(\frac{1}{T}\right) \int_0^T \sin^2\left(\frac{2\pi}{T}t\right) dt \quad (18)$$

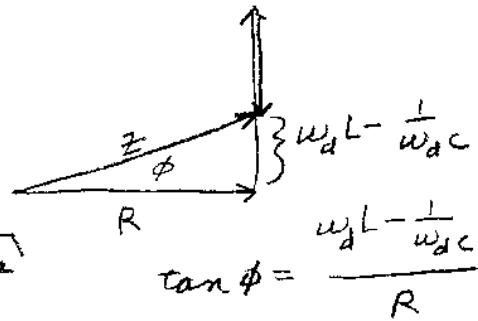
We will show in lecture that the average of  $\sin^2$  over one cycle equals 1/2. So we have for the average power:

Z:



$$X_L = \omega_d L$$

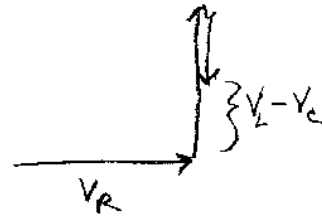
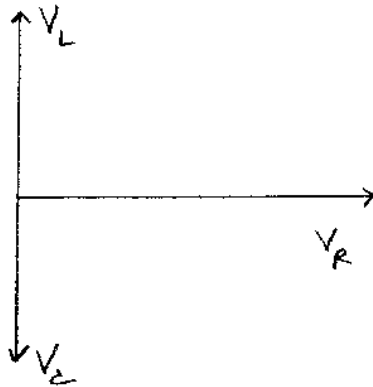
$$X_C = \frac{1}{\omega_d C}$$



$$Z = \sqrt{\left(\omega_d L - \frac{1}{\omega_d C}\right)^2 + R^2}$$

$$\tan \phi = \frac{\omega_d L - \frac{1}{\omega_d C}}{R}$$

Voltage:



$$V = \sqrt{V_R^2 + (V_L - V_C)^2}$$

$$P_{ave} = \frac{I_m^2 R}{2} \quad (19)$$

It is convenient to express the power in terms of the R.M.S. (Root Mean Square) value of the current (or voltage). The R.M.S. value means the square Root of the average (Mean) value of the Square of the function. For a sinusoidally varying function, the R.M.S. value equals 1/2 the value of the maximum. So,

$$I_{RMS} = \frac{I_m}{\sqrt{2}}$$

$$V_{RMS} = \frac{V_m}{\sqrt{2}}$$

in terms of the R.M.S. values,

$$P_{ave} = I_{RMS}^2 R$$

$$= I_{RMS} \frac{V_{RMS}}{Z} R$$

$$= I_{RMS} V_{RMS} \cos(\phi)$$

since  $\cos(\phi) = R/Z$ .

### Summary

In summary, there are three different types of circuit components. One type are resistive elements, which have  $V \propto I$  ( $V = RI$ ). Another type are inductive elements, which have  $V \propto dI/(dt)$  ( $V = L(dI/(dt))$ ). A third type are capacitive elements, which have  $I \propto dV/(dt)$  ( $I = (1/C)(dV/(dt))$ ). *If the current through the circuit element is  $I = I_m \sin(\omega_d t)$  then,*

$$V(t) = RI_m \sin(\omega_d t) \text{ (Resistive)}$$

$$V(t) = L\omega_d \cos(\omega_d t) \text{ (Inductive)}$$

$$V(t) = -\frac{1}{C\omega_d} \cos(\omega t) \text{ (Capacitive)}$$

In series, the voltages (and resistances) add like vectors (phasors) because the sum of two sinusoidals that have the same frequency is itself a sinusoidal with the same frequency.

## Maxwell's Equations

The equations for the electromagnetic fields that we have developed so far, in Phy133, are best expressed in terms of line and surface integrals:

$$\begin{aligned}\oint \oint \vec{E} \cdot d\vec{A} &= \frac{Q_{net}}{\epsilon_0} \quad (Coulomb) \\ \oint \oint \vec{B} \cdot d\vec{A} &= 0 \\ \oint \vec{E} \cdot d\vec{r} &= -\frac{d\Phi_B}{dt} \quad (Faraday) \\ \oint \vec{B} \cdot d\vec{r} &= \mu_0 I_{net} \quad (Ampere)\end{aligned}$$

Maxwell realized that the equations above are inconsistent with charge conservation. In particular, there is a problem with the equation referred to as Ampere's Law:

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 I_{net} \quad (20)$$

where  $I_{net}$  is the net current that passes through the path of the line integral  $\oint \vec{B} \cdot d\vec{r}$ .

We can demonstrate the problem by considering the magnetic field that is produced by a charging parallel plate capacitor. Let the capacitor have circular plates with a radius  $R$ , and a plate separation  $d$ . Let the wires that connect to the center of the plates extend to  $\pm\infty$ . See the figure on the adjacent page. Suppose the right plate has charge  $+Q(t)$  and the left plate a charge of  $-Q(t)$ . Suppose also that charge is flowing into the left plate and out of the right plate.

Suppose for a certain time period the current flowing into the plates is a constant, with value  $I$ . This current will produce a magnetic field that will circulate the wire. If  $d \ll R$ , the magnetic field a distance  $R$  from the wire will have a magnitude of  $B \approx \mu_0 I / (2\pi R)$ . Last quarter we derived this from the Biot-Savart Law. Since  $d \ll R$ , the magnetic field will also exist and have roughly this value at the edge (in the middle) of the capacitor. That is, a distance  $R$  from the axis and a distance  $d/2$  from one side.

Now, let's apply Ampere's law for a circular path, radius  $R$ , in the middle of the capacitor.



$$\begin{aligned}
\oint \vec{B} \cdot d\vec{r} &= \mu_0 I_{net} \\
B(2\pi R) &= \mu_0 I_{net} = 0 \\
B &= 0
\end{aligned}$$

The last line equals zero since there is no physical current going through the circular path. The direct surface of the path lies inside the capacitor (i.e. between the plates). There is no physical current flowing between the plates. If we choose a different surface, a curved surface (like a sock) that goes outside the plates and intersects the wire, then a physical current  $I$  would go through this "sock-like" surface. The way Ampere's law is stated above, it appears that one gets different results depending on the surface one chooses for "through the path".

Maxwell realized that there is a term missing from Ampere's law. For now, let's call this term  $M$  and reconsider Ampere's law with the surface being a flat circular surface through the middle of the capacitor.

$$\begin{aligned}
\oint \vec{B} \cdot d\vec{r} &= \mu_0 I_{net} + M \\
B(2\pi R) &= \mu_0 I_{net} + M \\
\frac{\mu_0 I}{2\pi R}(2\pi R) &= 0 + M \\
M &= \mu_0 I
\end{aligned}$$

So the missing term must equal  $\mu_0 I$ , as if the wire went right through the capacitor. But, there is no physical current between the capacitor plates. There is only an Electric field. We can express  $I$  in terms of the electric field between the plates. Since  $I = dQ/(dt)$ , we have

$$\begin{aligned}
M &= \mu_0 I \\
&= \mu_0 \frac{dQ}{dt} \\
&= \mu_0 \frac{d(\epsilon_0 A E)}{dt}
\end{aligned}$$

since the electric field between the capacitor plates is  $E = Q/(A\epsilon_0)$ , where  $A$  is the area of the plates. With this substitution, we have

$$\begin{aligned}
M &= \mu_0 \frac{d(\epsilon_0 AE)}{dt} \\
&= \mu_0 \epsilon_0 \frac{d(AE)}{dt} \\
M &= \mu_0 \epsilon_0 \frac{d\phi_E}{dt}
\end{aligned}$$

where  $\phi_E$  is the electric flux through the flat surface in the middle of the plates. Maxwell realized that if we add this term to Ampere's law, then the law is consistent with any surface that is chosen on the right side. The corrected equation is

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 I_{net} + \epsilon_0 \mu_0 \frac{d\phi_E}{dt} \quad (21)$$

For the charging parallel plate capacitor the problem is resolved. If one chooses the "sock" surface (outside the capacitor), there is a physical current that goes through the surface. The first term on the right is  $\mu_0 I$  and the second term is zero since there is no electric field outside the capacitor. If one chooses the flat surface that goes through the capacitor, the first term is zero since there is no physical current through this surface. However, the second term is  $\mu_0 I$  since  $\epsilon_0 (d\phi_E)/(dt)$  through this surface equals  $I$ . Maxwell named the quantity  $\epsilon_0 d\phi_E/(dt)$  as "displacement current".

Some things to note:

1. At the time, there was no experimental evidence for the displacement current term in  $\oint \vec{B} \cdot d\vec{r}$ . Maxwell used purely theoretical reasoning to justify its existence.
2. It might not be clear from our arguments how charge conservation enters the reasoning. When the field equations are expressed in differential form (junior level physics), it will become clearer. Without the displacement current Ampere's law would be  $\text{Curl}(\vec{B}) = \mu_0 \vec{J}$ . Since the Divergence of a Curl is always zero, we would have  $\text{Div}(\vec{J}) = 0$ . Charge conservation requires that  $\text{Div}(\vec{J}) = \partial\rho/(dt)$ . Since  $\text{Div}(\vec{E}) = \rho/\epsilon_0$ , the changing electric flux term restores charge conservation.
3. In our charging capacitor example, only one of the two terms were non-zero at a time. In general, both terms can be non-zero at the same time. We will discuss an example in class. That is, a magnetic field can be produced by a "real current" and the second "displacement current" piece at the same time.

4. You might wonder why the displacement current term was not discovered experimentally, as was Faraday's law. The reason it was not initially observed is that it is much smaller than the  $\mu_0 I$  piece since  $\epsilon_0$  is relatively small.
5. You might be concerned that the choice of surface may be important in Faraday's Law, and that perhaps a term is also missing from the equation  $\oint \vec{E} \cdot d\vec{l} = -(d\phi_B)/(dt)$ . There is not. Since there are no sources for  $\vec{B}$ , any valid surface you pick for applying Faraday's law will give the same answer.

We state here the complete set of source equations (in integral form) for the electric and magnetic fields:

$$\begin{aligned} \oint \oint \vec{E} \cdot d\vec{A} &= \frac{Q_{net}}{\epsilon_0} \text{ (Coulomb)} \\ \oint \oint \vec{B} \cdot d\vec{A} &= 0 \\ \oint \vec{E} \cdot d\vec{r} &= -\frac{d\Phi_B}{dt} \text{ (Faraday)} \\ \oint \vec{B} \cdot d\vec{r} &= \mu_0 I_{net} + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt} \text{ (Ampere + Maxwell)} \end{aligned}$$

These equations are referred to as "Maxwell's Equations" for the classical electrodynamic fields. The force equation  $F = q\vec{E} + q(\vec{v} \times \vec{B})$  relates the force to the fields. Maxwell added the missing piece in 1864. As we shall see in the next section, his contribution unified the electric-magnetic interaction with light (or more generally electromagnetic radiation). Radio, television, and wireless communication were made possible from an understanding of these equations once the displacement current piece was added.

### Electromagnetic Radiation

Let's consider solutions to Maxwell's equations in empty space, where there is no charge and no real currents. That is, in a region where all closed surface integrals do not contain any charge, and where there is no real current flowing through any closed path. In this case, Maxwell's equations become:

$$\oint \oint \vec{E} \cdot d\vec{A} = 0$$

$$\begin{aligned}
\oint \oint \vec{B} \cdot d\vec{A} &= 0 \\
\oint \vec{E} \cdot d\vec{r} &= -\frac{d\Phi_B}{dt} \\
\oint \vec{B} \cdot d\vec{r} &= \epsilon_0 \mu_0 \frac{d\Phi_E}{dt}
\end{aligned}$$

If  $\vec{E}$  and  $\vec{B}$  do not have any time dependence, then there are zeros on the right side of each equation. This is the "static" case in which all the free charge and currents don't change in time. In this case  $\vec{E}$  and  $\vec{B}$  can be calculated independent of each other. Let's try and find a simple time-dependent solution for  $\vec{E}$  in free space. The simplest case is an electric field that only points in one direction, say the "y" direction  $\hat{j}$ . Since the surface integral of  $\vec{E}$  for a closed surface equals zero,  $E_y$  cannot change in the y-direction. So,  $E_y$  can only change in a direction perpendicular to the y-axis. So, with little loss of generality, we can choose  $E_y$  to vary in the "x" direction. That is:

$$\vec{E} = E_y(x, t)\hat{j} \quad (22)$$

We now carry out  $\oint \vec{E} \cdot d\vec{r}$  around a small rectangle in the x-y plane. If the sides of the rectangle are  $\Delta x$  and  $\Delta y$ , we have

$$\begin{aligned}
\oint \vec{E} \cdot d\vec{r} &= -\frac{d\Phi_B}{dt} \\
E_y(x + \Delta x)(\Delta y) - E_y(x)(\Delta y) + 0 + 0 &= -\frac{d((\Delta x)(\Delta y)B_z)}{dt} \\
(\Delta y)(E_y(x + \Delta x) - E_y(x)) &= -(\Delta x)(\Delta y)\frac{dB_z}{dt} \\
(\Delta y)(\Delta x)\frac{dE_y}{dx} &= -(\Delta x)(\Delta y)\frac{dB_z}{dt} \\
\frac{dE_y}{dx} &= -\frac{dB_z}{dt}
\end{aligned}$$

Two of the legs on the left side are zero, since  $d\vec{r}$  is in the x-direction, but  $\vec{E}$  is in the y-direction. Only the legs in the y-direction contribute to the line integral on the left. On the right, the area for the magnetic flux is in the x-y plane, so only the z-component of  $\vec{B}$ ,  $B_z$  contributes. So we see that the z-component of the magnetic field is related to the y-component of the electric field.

$\oint \vec{E} \cdot d\vec{\ell} = \int_1^2 + \int_2^3 + \int_3^4 + \int_4^1$

$= 0 + \Delta y E_y(x+\Delta x) + 0 - \Delta y E_y(x)$

$= \Delta y \frac{(E_y(x+\Delta x) - E_y(x)) \Delta x}{\Delta x}$

$\oint \vec{E} \cdot d\vec{\ell} = \Delta y \Delta x \frac{\partial E_y}{\partial x}$

+y is out of the page

$\oint \vec{B} \cdot d\vec{\ell} = \int_1^2 + \int_2^3 + \int_3^4 + \int_4^1$

$= \Delta z B_z(x) + 0 - \Delta z B_z(x+\Delta x) + 0$

$= \Delta z (B_z(x) - B_z(x+\Delta x))$

$= -\Delta z \frac{(B_z(x+\Delta x) - B_z(x)) \Delta x}{\Delta x}$

$\oint \vec{B} \cdot d\vec{\ell} = -\Delta z \Delta x \frac{\partial B_z}{\partial x}$

The simplest solution will have  $\vec{B} = B_z \hat{k}$ . Since the surface integral of  $\vec{B}$  for a closed surface equals zero,  $B_z$  cannot change in the  $z$ -direction. Let's consider a solution that has  $B_z$  only a function of  $x$  and  $t$  as with  $E_y$ :

$$\vec{B} = B_z(x, t) \hat{k} \quad (23)$$

We now carry out  $\oint \vec{B} \cdot d\vec{r}$  around a small rectangle in the  $x$ - $z$  plane. If the sides of the rectangle are  $\Delta x$  and  $\Delta z$ , we have a similar result as with  $E_y$ :

$$\begin{aligned} \oint \vec{B} \cdot d\vec{r} &= \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \\ B_z(x)(\Delta z) - B_z(x + \Delta x)(\Delta z) + 0 + 0 &= \mu_0 \epsilon_0 \frac{d((\Delta z)(\Delta x)E_y)}{dt} \\ -(\Delta z)(B_z(x + \Delta x) - B_z(x)) &= \mu_0 \epsilon_0 (\Delta z)(\Delta x) \frac{dE_y}{dt} \\ -(\Delta z)(\Delta x) \frac{dB_z}{dx} &= \mu_0 \epsilon_0 (\Delta x)(\Delta z) \frac{dE_y}{dt} \\ \frac{dB_z}{dx} &= -\mu_0 \epsilon_0 \frac{dE_y}{dt} \end{aligned}$$

Once again, the  $z$ -component of the magnetic field is related to the  $y$ -component of the electric field. We can eliminate  $B_z$  (or  $E_y$ ) from these equations by differentiating the first equation with respect to  $x$ :

$$\frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial^2 B_z}{\partial x \partial t} \quad (24)$$

and the second equation with respect to  $t$ :

$$\frac{\partial^2 B_z}{\partial x \partial t} = -\mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2} \quad (25)$$

Combining these two equations gives

$$\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2} \quad (26)$$

This is a "wave equation". The solution is wave traveling in the  $\pm x$ -direction with a speed of  $v = 1/\sqrt{\epsilon_0 \mu_0}$ ,  $E_y(x, t) = f(x \pm vt)$  for any function  $f$ . A similar "wave equation" results for  $B_z$  by differentiating the first equation with respect to  $t$  and the second one with respect to  $x$ :

$$\frac{\partial^2 B_z}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 B_z}{\partial t^2} \quad (27)$$

The speed of these electromagnetic disturbances travel at a speed  $v = 1/\sqrt{\epsilon_0 \mu_0}$ . When the electromagnetic constants are plugged in, one gets the speed to be  $v = 1/\sqrt{(4\pi \times 10^{-7})(8.85 \times 10^{-12})} \approx 3 \times 10^8 \text{ m/s}$ . One has to believe that it is no accident that the two constants  $\epsilon_0$  and  $\mu_0$  are related to the speed of light as  $c = 1/\sqrt{\epsilon_0 \mu_0}$ , and that light is an electromagnetic disturbance (wave).

Maxwell's discovery that light is part of the electric and magnetic interaction led to a dramatic change in human life on earth. With a knowledge of the nature of light, we can understand how electromagnetic radiation can be produced and detected. We can see from the equations above that the traveling wave requires that both  $\vec{E}$  and  $\vec{B}$  depend on time. That is, radiation will not be produced from stationary charge sources or steady currents. Even a charged object moving at a constant velocity will not radiate, since the object will be at rest in an inertial frame moving (at constant velocity) with the object. So, in order to produce electromagnetic radiation (classically from Maxwell's equations) *charged objects need to be accelerated*. In order to sustain the radiation, charged objects, i.e. electrons, will need to move in a circle or "back and forth". Thus, most of the radiation produced classically can be characterized by its frequency. Although the radiation might not be exactly sinusoidal in time (AM is modulated in amplitude and FM in frequency), we can refer to the radiation in terms of its frequency. Below we list the names we assign to different parts electromagnetic spectrum.

Name	frequency $f$ (Hz)	wavelength $\lambda$	energy $E$	source
Detection:	electric circuits	interference effects	Photo-absorption	
radio/TV	$10^3 - 10^6 \text{ Hz}$			electrons oscillating in wires (AC)
microwaves	$10^8 - 10^9 \text{ Hz}$	10 – 300 cm		accelerating electrons Molecular Rotations
Infra-Red		$3 \times 10^{-5} \text{ m}$		Molecular Vibrations
visible		400 – 700 nm	1 – 3 eV	Atomic Transitions
Ultra-Violet		10 – 400 nm	3 – 100 eV	Atomic Transitions
X-rays		$\approx 1 \text{ nm}$	1 – 100 KeV	large Z inner electron and nuclear transitions
gamma			> 100 KeV	Nuclear transitions

In the table above,  $\lambda = c/f$  and  $E = hf$ . In the next course, Phy235, you will learn about the energy of the radiation. We can directly measure frequencies if they are below around  $10^{10}$  Hz. We can measure wavelength from around 10 cm to around 1 nm using interference effects. Photon energies can be measured if they are greater than a few electron volts.

Some things to note about our derivation so far:

1. The solution with  $\vec{E} = E_y(x,t)\hat{j}$  and  $\vec{B} = B_z(x,t)\hat{k}$  are called "plane wave" solutions. The electric field vector is the same at every point in the entire y-z plane.  $\vec{E}$  only varies in space in the "x-direction". Similarly, the magnetic field vector,  $\vec{B}$ , only varies in space in the "x-direction".
2. Note that  $\vec{E}(x,t)$  and  $\vec{B}(x,t)$  are coupled. There is no solution for electromagnetic radiation that only has an electric field vector, nor only a magnetic field vector. To have a solution  $E_y(x,t)$ , one also needs  $B_z(x,t)$ . The coupling is perpendicular. That is, a time changing  $E_y$  produces a magnetic field in the z-direction. A time changing  $B_z$  produces an electric field in the y-direction. The choice of the y-axis for  $\vec{E}$  was arbitrary. Whatever direction we would have chosen for  $\vec{E}$ , both the radiation direction and the coupled  $\vec{B}$  field would have been perpendicular to  $\vec{E}$ .
3. Both  $E_y(x,t)$  and  $B_z(x,t)$  must have the same space and time variation. That is if  $E_y(x,t) = E_0g(x \pm ct)$ , then  $B_z(x,t) = B_0g(x \pm ct)$ . Since  $E_y$  and  $B_z$  are coupled,  $E_0$  and  $B_0$  are related to each other. To find the relationship, we carry out the derivatives:

$$\begin{aligned}\frac{dE_y}{dx} &= -\frac{dB_z}{dt} \\ E_0g' &= -B_0(\pm c)g' \\ E_0 &= \mp cB_0\end{aligned}$$

by the chain rule. So the maximum value of the electric field  $E_y$  equals  $c$  times the maximum value of  $B_z$ .

4. Electromagnetic radiation usually varies sinusoidally in time and space. So a common form for  $g(x \pm ct)$  is



$$\begin{aligned}
 E_y &= E_0 \sin(kx \pm \omega t) \\
 B_z &= B_0 \sin(kx \pm \omega t)
 \end{aligned}$$

where  $k = 2\pi/\lambda$ ,  $\omega = 2\pi f$ , and  $E_0 = cB_0$ . Here  $f$  is the frequency of the radiation.

5. Note that for " $(kx - \omega t)$ " the radiation travels in the  $+x$  direction and  $B_z = +E_y/c$ . For " $(kx + \omega t)$ " the radiation travels in the  $-x$  direction and  $B_z = -E_y$ . Thus the direction of the radiation is always in the  $\vec{E} \times \vec{B}$  direction.
6. Experimental evidence for electromagnetic radiation was performed by Hertz in 1887. He produced the radiation by opening and closing a circuit with a large coil. He detected the radiation using another coil with a gap. He was able to verify many of the properties of the radiation predicted by Maxwell's equations.
7. As we showed in our last course (Phy133), if we choose a different unit for charge, both  $\epsilon_0$  and  $\mu_0$  each change, but the product  $\epsilon_0\mu_0$  does not. The product  $\epsilon_0\mu_0$  has units of  $time^2/length^2$ , and is independent of our choice of units for charge.

We now consider the energy and momentum properties of the electromagnetic radiation, and its polarization.

### *Energy of electromagnetic radiation*

Using the classical picture of electromagnetic radiation, we can find an expression for the energy density of the radiation. Last quarter we showed that the energy density of the electric field is given by:

$$U_E = \frac{\epsilon_0 E^2}{2} \tag{28}$$

We also showed that the energy density of the magnetic field is given by:

$$U_B = \frac{B^2}{2\mu_0} \tag{29}$$

So the complete energy density of electromagnetic radiation is

$$\begin{aligned}
U &= U_E + U_B \\
&= \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}
\end{aligned}$$

Since  $|B| = |E|/c$ , we can combine the two terms. For the rest of the discussion, let's take the radiation to have sinusoidal space and time dependence, which is the best basis to work in. That is,  $\vec{E} = E_0 \sin(kx - \omega t) \hat{j}$  and  $\vec{B} = B_0 \sin(kx - \omega t) \hat{z}$ , where  $B_0 = E_0/c$ . In this case

$$\begin{aligned}
U(t) &= \left( \frac{\epsilon_0 E_0^2}{2} + \frac{B_0^2}{2\mu_0} \right) \sin^2(kx - \omega t) \\
U(t) &= \left( \frac{\epsilon_0 E_0^2}{2} + \frac{E_0^2}{2\mu_0 c^2} \right) \sin^2(kx - \omega t) \\
&= \epsilon_0 E_0^2 \sin^2(kx - \omega t)
\end{aligned}$$

where we have used  $c^2 = 1/(\epsilon_0 \mu_0)$ . As we can see, the energy density of the electromagnetic field varies sinusoidally in space and time. It is most convenient to consider the average energy density. That is, the energy density averaged over time for a fixed location in space. We showed before that the average of  $\sin^2(\omega t + \theta)$  over one complete cycle is  $1/2$ . Therefore, the time averaged energy density is

$$U_{ave} = \frac{\epsilon_0 E_0^2}{2} \tag{30}$$

$U_{ave}$  is usually expressed in terms of  $E_0$ , but we could also have expressed  $U_{ave}$  as  $U_{ave} = B_0^2/(2\mu_0)$ . Note that  $U_E$ ,  $\epsilon_0 E_0^2/2$ , and  $U_B$ ,  $B_0^2/(2\mu_0)$  have the same magnitude since  $B_0^2 = E_0^2/c^2$  and  $c^2 = 1/(\epsilon_0 \mu_0)$ .

A useful quantity to consider is the energy of the EM radiation per area per unit time. This quantity is referred to as the intensity of the radiation. Since the radiation travels at the speed  $c$ , the energy passing through an area  $A$  in a time  $\Delta t$  is

$$energy = UA(c\Delta t) \tag{31}$$

since  $U$  is the energy per volume and a volume of  $A(c\Delta t)$  passes through the area in a time  $\Delta t$ . So,

$$\begin{aligned}
I &= \frac{\text{energy}}{A(\Delta t)} \\
&= cU \\
&= c\epsilon_0 E_0^2 \sin^2(kx - \omega t)
\end{aligned}$$

We can express the expression for  $I$  in terms of the magnetic field

$$\begin{aligned}
I &= c\epsilon_0 E_0 (cB_0) \sin^2(kx - \omega t) \\
&= c^2 \epsilon_0 E_0 B_0 \sin^2(kx - \omega t) \\
&= \frac{E_0 B_0}{\mu_0} \sin^2(kx - \omega t)
\end{aligned}$$

Since the wave travels in a direction perpendicular to both  $\vec{E}$  and  $\vec{B}$  we can define an "intensity vector" with a magnitude of  $I = EB/\mu_0$  in the direction of the radiation as:

$$\vec{S} \equiv \frac{1}{\mu_0} \vec{E} \times \vec{B} \quad (32)$$

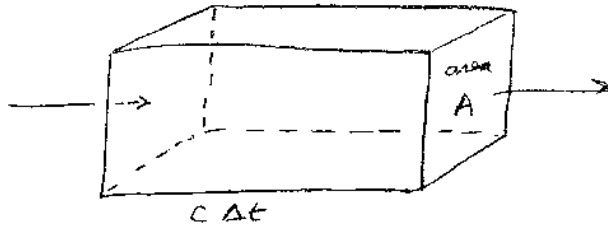
This vector was first investigated by Poynting, and is called "Poynting's Vector". Its magnitude equals the energy/area/time (intensity), and it points in the direction of the energy flow.

The Poynting vector equals the electromagnetic energy that is transported per area, per time. Although we showed this property here for electromagnetic radiation, it can be shown to be always valid. In your next EM course, Phy314, you will show, using vector calculus that the rate at which electromagnetic work  $W$  is done within a volume equals the Poynting vector integrated around the surface.

$$\frac{dW}{dt} = \int \vec{E} \cdot \vec{J} dV = -\frac{d}{dt} \int \frac{1}{2} (\epsilon_0 E^2 + B^2/\mu_0) dV - \oint \frac{\vec{E} \times \vec{B}}{\mu_0} \cdot d\vec{A} \quad (33)$$

We leave this proof for your next course on electromagnetism, but mention it here so you will appreciate the significance of Poynting's Vector. We will do examples in class.

A distinguishing feature derived from Maxwell's equations is that *the energy of electromagnetic radiation (i.e. intensity) is proportional to the electric field squared,  $E^2$* . That is, brighter light will have a larger electric field. The same relationship



$$\text{Energy passing through} = \frac{\text{Energy}}{\text{Vol}} (c\Delta t) A$$

$$= U c (\Delta t A)$$

$$I = \frac{\text{Energy}}{\text{area time}} = U c$$

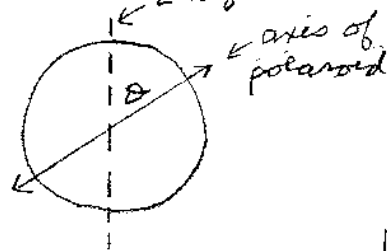
$$= c \epsilon_0 E_0^2 \sin^2(kx - \omega t)$$

$$= \frac{1}{\mu_0} E B$$

$$\vec{S} \equiv \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

Linear Polaroids:  $\vec{E}$  before

$\vec{E}_i$  Before  
 $|\vec{E}_i| = E_0$



$\vec{E}_s$  after  
 $|\vec{E}_s| = E_0 \cos^2 \theta$

$$I_s = I_0 \cos^2 \theta$$

between wave amplitude and energy is true for sound waves, water waves, and waves in general that have a medium, i.e. mechanical waves. As you will see next quarter, the photoelectric effect cannot be understood from the classical model of electromagnetic radiation presented here.

### *Polarization*

In the solution we just derived for electromagnetic radiation, we chose the electric field to point in the  $\hat{j}$  direction. If we would have chosen  $\vec{E}$  to point in the  $\hat{k}$  direction, then  $\vec{B}$  would have been in the  $-\hat{j}$  direction and the wave would still have propagated in the  $+x$ -direction. The direction of propagation of the radiation is in the  $\vec{E} \times \vec{B}$  direction. So,  $\vec{E}$  can point anywhere in the  $y$ - $z$  plane and the radiation can still travel in the  $+x$  direction.

If  $\vec{E}$  points in either the  $+\hat{j}$  or  $-\hat{j}$  direction, the radiation is said to be linearly polarized in the "y-direction". If  $\vec{E}$  points in either the  $+\hat{z}$  or  $-\hat{z}$  direction, the radiation is said to be linearly polarized in the "z-direction". Any superposition of these two solutions is also linearly polarized in the direction of the  $\vec{E}$  field. There are two linear polarization states for electromagnetic radiation. For example, for radiation that propagates in the  $+x$  direction, and is linearly polarized at an angle  $\theta$  with respect to the  $y$ -axis, the electric field is

$$\vec{E}(x, t) = E_0(\cos(\theta)\hat{j} + \sin(\theta)\hat{k})\sin(kx - \omega t) \quad (34)$$

where  $E_0$  is the magnitude of the electric field. The corresponding magnetic field is

$$\vec{B}(x, t) = B_0(\cos(\theta)\hat{k} - \sin(\theta)\hat{j})\sin(kx - \omega t) \quad (35)$$

where  $B_0 = E_0/c$ . Another interesting combination of the two polarization states is the following:

$$\vec{E}(x, t) = E_0\hat{j}\cos(kx - \omega t) + E_0\hat{k}\sin(kx - \omega t) \quad (36)$$

with corresponding magnetic field

$$\vec{B}(x, t) = B_0\hat{k}\cos(kx - \omega t) - B_0\hat{j}\sin(kx - \omega t) \quad (37)$$

where  $B_0 = E_0/c$ . For this combination, the electric field vector rotates (via the right hand) in space and time. This type of polarization is termed right-handed "circularly polarized" radiation. For left-handed circularly polarized light, the electric field and magnetic fields are

$$\begin{aligned}\vec{E}(x, t) &= E_0 \hat{j} \cos(kx - \omega t) - E_0 \hat{k} \sin(kx - \omega t) \\ \vec{B}(x, t) &= B_0 \hat{k} \cos(kx - \omega t) + B_0 \hat{j} \sin(kx - \omega t)\end{aligned}$$

One could also have elliptically polarized radiation. There are two independent polarization states for electromagnetic radiation. One can express the polarization in terms of two linearly polarization states, in terms of two circularly polarization states, or in terms of any two independent basis states.

Light from an incandescent source, i.e. light bulb, has equal amount of both polarizations, and is "unpolarized". There exist polaroids that allow only one type of polarization state pass through. A linear polaroid will have a polarization axis. Radiation that is polarized along the polarization axis will pass through, and radiation that is polarized perpendicular to the polarization axis will not. If unpolarized radiation "hits" a linear polaroid, then the intensity is reduced by a factor of 1/2. The polarization of the transmitted radiation will be along the axis of the polaroid.

Consider linearly polarized radiation that "hits" a linear polaroid. Let  $\theta$  be the angle between the direction of the polarization of the radiation and the axis of the polaroid. Then, the electric field that passes through the polaroid will be reduced by a factor of  $\cos(\theta)$  after passing through the polaroid. The intensity of the radiation will be decreased by a factor of  $\cos^2(\theta)$ . The polarization of the transmitted radiation will be along the axis of the polaroid. We will do some nice examples in lecture with polaroids.

This concludes the first third of our course. In the next third we will continue with our investigation of electromagnetic radiation, in particular light. The main topics will be geometric and physical optics.