Notes on the Magnetic Interaction

In the last sections, the main interaction we considered was the electrostatic: Coulomb's Law for point charges and the principle of superposition. Why did we call Coulomb's Law the electro**static** force? This is because the force does not **depend** on the particle's velocity. Whether the charged particles are moving or not the force is the same: kq_1q_2/r^2.

In this section, we shall see that if the charged particles are both moving, then there is an additional force between them. We could call this force the "velocity dependent part of the electric force", but this is too wordy. The name we give it is the **magnetic force**, and together with the electrostatic force, the complete force is called the **electromagnetic interaction**.

We first present some properties of permanent magnetics, then develop the properties of the electromagnetic interaction.

Permanent Magnets

The magnetic force was discovered long before it was realized that its source was moving charges. This is because permanent magnets exert forces on each other even though the magnet is not moving nor has any charge on it. We list some properties of permanent magnets below:

1. Magnets always come with two "poles". No one has yet to find a magnet with only one pole, a magnetic monopole. Whenever there is a north pole, there always is a south pole.

2. Like poles repel, and opposite poles attract.

3. If a magnet is cut in half, then two new magnets are formed. That is if one starts with one North and South pole, and cut it in half, then one gets two smaller magnets each with their own North and South pole pair.

You might wonder what would happen if you kept dividing a magnet in half over and over again. Eventually you get to a single atom. Atoms themselves are little magnets with a north and south pole. What about the constituents of the atom: the protons, neutrons and electrons? The proton, neutron and electron are each little magnets as well. This is somewhat perplexing, since as far as we know the electron has very little, if any, size. The subject of quantum mechanics is needed to understand the magnetic properties of these particles. In this class we won't get that far, but will cover the developments of electromagnetism up to around 1860. We begin with one of the most important discoveries in science.

Sources of Magnetism

Many discoveries in science are by accident, and in 1820 a very important accidental discovery changed the way the world was to be. The place was Hans Oerstead's classroom. Oerstead produced a large current in a wire that was in the vicinity of a magnetic compass. When the current flowed in the wire, he noticed that the magnetic compass needle moved! This was the first evidence that electric currents produced magnetic fields. In summary, he discovered that **electric currents produce magnetic fields**, or stated in microscopic terms: **moving charge produces magnetic fields**.

One cannot understate the importance of this discovery. Oerstead's discovery showed that two "supposedly different" forces were somehow related to each other. Over the next 40 years scientists sorted out the connections and unification of these two forces: electric and magnetic. It was the beginning of a quest to see if all the fundamental forces of nature were somehow related to each other. From 1960-1980, the "weak" interaction was unified with the electric and magnetic. Current experiments are examining if the strong and/or graviational interactions are also part of the electromagnetic-weak interaction. In addition to these philosophical implications, Oerstead's discovery eventually lead to many practical applications. Before we can understand the applications, we need to understand the "physics" of the magnetic field produced by an electric current.

Initial Definition of the Magnetic Field

We will initially define the magnetic field $\vec{B}$ at a point "P" in space in terms of how it affects a little magnetic probe when the probe is located at "P". If we place our little magnetic probe at "P", it will allign itself along the direction of the magnetic field. The strength of the magnetic field is related to how strong the probe alligns itself with the field. In lecture we will use the "magnaprobe" to determine the magnetic field from different sources.

Later we will define the magnetic field in terms of the force it produces on a small "point" particle, but for now we will define it in terms of its effect on a small magnetic probe.

Quantitative Description of the Magnetic Field

In lecture we will connect a power supply to a wire and produce an electrical current in the wire. Then, using our little "magnaprobe" we will see that the magnetic field produced by this "current carrying wire" *goes around the wire*. This might seem a bit strange, but lets try to understand why it is so. As physicists, we first start with the simplest situation and try and determine what quantites the magnetic field could depend upon.

Let's first consider a small piece of the wire and pose the following question: What is the magnetic field $\Delta\vec{B}$ at a point "P" located a position $\vec{r}$ from the small segment of the wire $\vec{\Delta l}$ that carries a current I? The vector $\vec{r}$ is a vector from the small wire segment to the point "P" (where we want to know the magnetic field $\Delta\vec{B}$).

Let the current in the wire be I, and the length of the wire be Δl. The direction of the current will be important, so we will designate the direction of the current by the direction of Δl. This can be done by making Δl a vector: $\vec{\Delta l}$.

What can the magnitude of the magnetic field $|\Delta\vec{B}|$ depend on? The only quantities available are: I, $\vec{r}$ and $\vec{\Delta l}$. How could the magnitude depend on them: A good guess is the following: if the current I is doubled, then the magnitude of the magnetic field should double. If the length of the small line segment is doubled, then so should the magnitude of $\Delta\vec{B}$:

$$|\Delta\vec{B}| \propto I|\vec{\Delta l}| \tag{1}$$

How might the magnitude of $\Delta\vec{B}$ depend on the distance from the small wire segment? It probably decreases with distance, but does it fall off as $1/r$ or $1/r^2$ or some other power? We are tempted to be guided by Newtonian gravity and Coulomb's law and suppose that the inverse square law applys here also. So a good guess would be

$$|\Delta\vec{B}| \propto \frac{I|\vec{\Delta l}|}{r^2} \tag{2}$$

where $r \equiv |\vec{r}|$. I should mention that this equation has been tested by experiment, so our guessing was good.

Now we need to consider the direction of the "little" magnetic field $\Delta\vec{B}$. You might guess that it points radially away from (or towards) the wire segment, like the electrostatic force. The direction of $\vec{r}$ should have some effect on the direction of the magnetic field. However, the source for the magnetic field is different than the source for the electrostatic or gravitational forces. In addition to $\vec{r}$, another vector enters the picture. The **source** of magnetism itself **has a direction**: the direction of the current in the wire. If there is no current (i.e. no direction) there is no magnetic field. Remember that the magnetic field is created by moving charges (current). So the direction of $\vec{\Delta l}$ must play a role in determining the direction of $\vec{\Delta B}$.

There are three vectors involved in determining the magnetic field produced by a small piece of current carrying wire: $\vec{\Delta l}$ which points in the direction of the current, $\vec{r}$ the vector from the wire segment to the location, and $\vec{\Delta B}$ the magnetic field at the location in question. The only mathematical operation (with the correct mathematical properties) that combines two vectors to produce a third vector is the *vector cross product*. Consequently, the only acceptable relationship is:

$$\vec{\Delta B} \propto \frac{I}{r^2}\,(\vec{\Delta l} \times \hat{r}) \tag{3}$$

where $\hat{r} \equiv \vec{r}/r$. This is a fairly complicated equation. We restate what the terms mean: $\vec{\Delta B}$ is the magnetic field at the point "P" produced by a small piece of wire which has a current I. $\vec{\Delta l}$ is a vector whose magnitude is the length of the "small piece of wire" and whose direction points in the direction of the current. $\vec{r}$ is a vector from the piece of wire to the point "P".

Note that because of the cross product, the direction of $\vec{\Delta B}$ will be perpendicular to both $\vec{\Delta l}$ and $\vec{r}$. Thus, the magnetic field will circulate a straight wire, a phenomena that was demonstrated in lecture.

We can replace the proportionality sign with an equal sign and a constant:

$$\vec{\Delta B} = \frac{\mu_0}{4\pi}\frac{I}{r^2}\,(\vec{\Delta l} \times \hat{r}) \tag{4}$$

where μ_0 is a constant equal to $4\pi \times 10^{-7}$ Tm/A. We will discuss units later on in the quarter. For now accept the fact that $\mu_0 = 4\pi \times 10^{-7}$ Tm/A. The "T" stands for Tesla, which is a unit for magnetic field. We need to determine what the units for the magnetic field are, which we will do shortly when we examine the force on a particle possessing charge.

If you don't like using the unit vector $\hat{r}$, you can write the above equation in terms of $\vec{r}$, since $\hat{r} = \vec{r}/r$. In terms of $\vec{r}$ we have

$$\vec{\Delta B} = \frac{\mu_0}{4\pi}\frac{I}{r^3}\,(\vec{\Delta l} \times \vec{r}) \tag{5}$$

Taking the magnitude of the right side of the equation above, we obtain an expression for the magnitude of $\vec{\Delta B}$:

$$|\vec{\Delta B}| = \frac{\mu_0}{4\pi}\frac{I|\vec{\Delta l}|sin(\theta)}{r^2} \tag{6}$$

where θ is the angle between $\vec{\Delta l}$ and $\vec{r}$. Since the sin function is maximized when $\theta = 90°$, the magnetic field is maximized at positions perpendicular to the wire segment. Since $sin(0) = sin(180°) = 0$ the wire segment does not produce any magnetic field at points located on its axis. The above formula is called the "Biot-Savart" Law in honor of the scientists who discovered it.

The Biot-Savart Law relates the magnetic field at a point in space produced by a small segment of a wire which carries a current. What if there are more than one small segments of wire which carry current (there always is). Then, to find the net magnetic field $\vec{B}$ at a point in space we might suspect that one would just add all the vectors $\vec{\Delta B_i}$ from each i'th wire segment. This is the **principle of superposition** applied to the magnetic field where the sources are segments of wire. Although we hope the superposition principle is valid for the "current sources" of the magnetic field, the principle needs to be verified by experiment. Experiments show that it is. Thus, now we have a method of calculating the magnetic field produced by any finite wire which has an electrical current flowing through it:

1. Divide the wire up into small segments, $\vec{\Delta l_i}$. The segments are labeled by the index i.

2. Use the Biot-Savart law to determine the magnetic field $\vec{\Delta B_i}$ at the point "P" from each of the segments.

3. Add up all the $\vec{\Delta B_i}$ to find the net magnetic field $\vec{B}$ at the point "P" due to the whole wire (i.e. apply the principle of superposition). In adding up the contributions from all the segments, we take the limit as the segment length $|\vec{\Delta l}|$ goes to zero. This will result in an integral which we need to evaluate.

Note that the above method is the same procedure we used for the electric field: Divide the object into small point sources, determine $\vec{E}$ from each point source in the object, and add up the contributions (i.e. integrate) over the whole object. We apply the above procedure to two common current distributions: a straight length of wire of finite size, and a circular loop of wire.

Magnetic field produced by a straight wire

Consider a wire segment that lies on the y-axis. The lower end starts at $y = -b$, and the upper end is at $y = a$. Suppose there is a current I that flows in the wire. Lets find the magnetic field at a point "P" on the x-axis located a distance d from the origin (d,0).

We start by dividing up the wire in to small segments of length Δy. The i'th segment will be located at $y_i = i * \Delta y$ where i goes from $-b/(\Delta y)$ to $a/(\Delta y)$. The distance from the i'th segment to the point "P" is $r = \sqrt{y_i^2 + d^2}$. The Biot-Savart Law gives for the magnitude of the magnetic field $\Delta\vec{B}_i$:

$$|\Delta\vec{B}_i| = \frac{\mu_0 I}{4\pi}\frac{\Delta y\ sin(\theta)}{r^2} \tag{7}$$

where θ is the angle between the direction of the wire segment and the point "P". In terms of the geometry, $sin(\theta) = d/r$.:

$$|\Delta\vec{B}_i| = \frac{\mu_0 I}{4\pi}\frac{\Delta y\ d}{r^3} \tag{8}$$

In terms of the y_i, we have

$$|\Delta\vec{B}_i| = \frac{\mu_0 I}{4\pi}\frac{\Delta y\ d}{(y_i^2 + d^2)^{3/2}} \tag{9}$$

Note: the direction of all the $\Delta\vec{B}_i$ are in the negative z direction, i.e. along $(-\hat{k})$. Adding up all the $\Delta\vec{B}_i$ and taking the limit as $\Delta y \to 0$ we have

$$|\vec{B}| = \frac{\mu_0 I d}{4\pi}\int_{-b}^{a}\frac{dy}{(y^2 + d^2)^{3/2}} \tag{10}$$

This integral can be solved using the trig substitution $y = d\ tan(\phi)$. With this substitution, $dy = d\ sec^2\phi\ d\phi$. This substitution also gives $y^2 + d^2 = d^2 sec^2\phi$. Upon substitution, the integral is just an integral over $cos(\theta)$:

$$|\vec{B}| = \frac{\mu_0 I}{4\pi d}\int_{-\phi_b}^{\phi_a} cos\phi\ d\phi \tag{11}$$

Solving the integral and taking the limits yields:

$$|\vec{B}| = \frac{\mu_0 I}{4\pi d}(sin(\phi_a) + sin(\phi_b)) \tag{12}$$

In terms of a and b the expression becomes

$$|\vec{B}| = \frac{\mu_0 I}{4\pi d}(\frac{a}{\sqrt{a^2 + d^2}} + \frac{b}{\sqrt{b^2 + d^2}}) \tag{13}$$

If the wire is of infinite length, the result takes on a simple form. In this case, ϕ_a and ϕ_b are both 90°. Since $sin(90°) = 1$, we have

$$|\vec{B}| = \frac{\mu_0 I}{2\pi d} \tag{14}$$

Here we see that the strength of the magnetic field decreases as the inverse of the perpendicular distance d to the wire. This $1/d$ decrease was also the case for the magnitude of the electric field produced by an infinitely long uniformly charged line (or rod) of charge. In infinitly long source essentially removes one dimension from the source distribution geometry. Since the circumference of a circle increases with radius to the first power, the force fields decrease with $1/d$. We will use this property of the magnetic field to reformulate the Biot-Savart Law via closed path line integrals: Ampere's Law.

Magnetic Field produced by a circular current loop

A common wire shape is a circular loop. Let's use the Biot-Savart Law to find the magnetic field on the axis of a circular loop of wire that has a current I flowing through it. Let the radius of the loop be R, the center of the loop be at the origin, and the loop lie in the y-x plane. The x-axis is thus the axis through the center of the loop Suppose the point "P" is located on the axis of the loop (i.e. the x-axis) a distance x from its center.

We use the same method as before, we first "chop" up the wire into small wire segments of equal length $|\Delta \vec{l}|$. The magnitude of the magnetic field $|\Delta \vec{B}|$ from any of the small segments of wire is

$$|\Delta \vec{B}| = \frac{\mu_0 I}{4\pi} \frac{|\Delta \vec{l}|}{x^2 + R^2} \tag{15}$$

The direction of $\Delta \vec{B}$ is not along the x-axis, but rather at an angle ϕ with respect to the axis. After integrating around the circular loop, all components of $\Delta \vec{B}$ perpendicular to the x-axis cancel out. Thus, the only component that "survives" is the one along the x-axis ΔB_x:

$$\begin{aligned} \Delta B_x &= \frac{\mu_0 I}{4\pi} \frac{|\Delta \vec{l}|}{x^2 + R^2} cos(\phi) \\ &= \frac{\mu_0 I}{4\pi} \frac{|\Delta \vec{l}|}{x^2 + R^2} \frac{R}{\sqrt{x^2 + R^2}} \end{aligned}$$

When the ΔB_x are summed (or integrated) x and R do not change. Thus, one only needs to sum up the $|\vec{\Delta l}|$ which is the circumference of the loop: $2\pi R$:

$$\Delta B_x = \frac{\mu_0 I}{4\pi} \frac{2\pi R}{x^2 + R^2} \frac{R}{\sqrt{x^2 + R^2}} \tag{16}$$

Collecting terms we have:

$$B_x = \frac{\mu_0 I R^2}{2(x^2 + R^2)^{3/2}} \tag{17}$$

You will test the x dependence of this equation in your laboratory class. If you find it to follow the above formula, you will be demonstrating the validity of the Biot-Savart Law for the current loop.

To find the magnetic field at the center of the loop, we set $x = 0$. Thus at the center of a circular loop we have

$$B_{center} = \frac{\mu_0 I}{2R} \tag{18}$$

with the direction perpendicular to the plane of the loop.

Magnetic Field on the axis of a solenoid

Another interesting current distribution is that of a solenoid. A solenoid is essentially wire coiled around a cylinder. We can use the results of the last section to find the magnetic field on the axis of a solenoid. Suppose the solenoid has a radius R and n turns of wire per length. Lets use our previous result to calculate the magnetic field on the axis inside the solenoid. Let the axis of the solenoid lie on the x-axis, and suppose the solenoid starts at $x = -b$ and extends to $x = a$, and we will calculate the magnetic field at the origin. If the current flows counter-clockwise looking in the "-x" direction, then the magnetic field at the origin will point in the +x direction.

Consider first the contribution to the magnetic field, ΔB_x, at the origin due to a small slice of the solenoid located at position x and of thickness Δx. The amount of current in the slice of thickness Δx is $\Delta I = n\Delta x$. From the previous section we have:

$$\Delta B_x = \frac{\mu_0 R^2 n \, \Delta x}{2(x^2 + R^2)^{3/2}} \tag{19}$$

To find the net magnetic field at the origin, we need to add up the ΔB_x from each thin slice of the solenoid. This leads to the integral

$$B_x = \mu_0 R^2 In \int_{-b}^{a} \frac{dx}{2(x^2 + R^2)^{3/2}} \tag{20}$$

We have done this integral before when we calculated the magnetic field produced by a finite straight current carrying wire. The trick was to substitute $x = Rtan\phi$. Then $dx = Rsec^2\phi$, and $(x^2 + R^2) = R^2 sec^2\phi$. After substitution the integral becomes

$$B_x = \frac{\mu_0 In}{2} \int_{-\phi_b}^{\phi_a} cos\phi \; d\phi \tag{21}$$

Evaluating the integral gives

$$B_x = \frac{\mu_0 In}{2}(sin(\phi_a) + sin(\phi_b)) \tag{22}$$

B_x can be written In terms of a and b:

$$B_x = \frac{\mu_0 In}{2}(\frac{a}{\sqrt{a^2 + R^2}} + \frac{b}{\sqrt{b^2 + R^2}}) \tag{23}$$

An interesting case is an infinite solenoid, that is when $a \to \infty$ and $b \to \infty$. Then, one obtains

$$B_x = \mu_0 In \tag{24}$$

for points on the axis of the infinite solenoid. Note here that the length of the solenoid and hence the total number of turns is infinite, but the number of turns per length, n, is finite.

Ampere's Law

In the last section we discussed the Biot-Savart Law, and applied it to some examples. As discussed, one can use the Biot-Savart law to find the magnetic field at any point in space that is produced by any current configuration. To do this, one would follow the same method we have used: slice up the wires into small segments, determine $\Delta\vec{B}_i$ from each of the segments, take the limit as the segments approach zero and integrate over all the wires. Although this can be a tedious process, computers can help and in principle one could solve any problem involving currents in wires.

We could continue solving other current set-ups, but it is more interesting to examine if there are any global properties of magnetic fields. We will see that a line (or path) integral of the magnetic field for a closed path yields simple solutions. This result will help us determine magnetic fields for some symmetric wire configurations.

The magnitude of the magnetic field produced by an infinitely long wire is $B = \mu_0 I/(2\pi d)$, where d is the perpendicular distance to the wire. The direction of the magnetic field is circular, around the wire. The denominator, $2\pi d$, reminds us of the circumference of a circle. Consider a circle such that the wire passes through its center and whose plane is perpendicular to the wire. If one were to calculate the line integral (or path integral) of $\vec{B} \cdot d\vec{r}$ for a closed path around this circle, one obtains

$$\oint \vec{B} \cdot d\vec{r} = \frac{\mu_0 I}{2\pi d}(2\pi d) \tag{25}$$

The result is simple since the magnitude of the magnetic field is constant, $B = \mu_0 I/(2\pi d)$, and the direction of the magnetic field is parallel to the path. In this special case, the circular path with the wire at the center, the path integral is simply the magnitude of the magnetic field times the path length. Note that the factor $2\pi d$ cancels, and we are left with

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 I \tag{26}$$

for a circular path of any size. Since the magnetic field decreases as $1/d$ and the path length increases as $2\pi d$, the radius of the circle cancels out. We will show in lecture that this results hold for any path that goes around the infinitely long wire.

You might be wondering if the direction of the current and/or the direction that the path is traversed in the closed path integral. Yes it does. The convention is that if your right thumb points in the direction of the current (for the infinite wire) the magnetic field circles in the direction of your fingers. Therefore, if the *fingers of your right hand curl in the direction of the path of the line integral*, your *thumb will point in the positive direction for current.* If the current flows in the opposite direction as your thumb, then the line integral will be negative. That is, the path is going in the opposite direction to $\vec{B}$.

What happens if more than infinite wire (carrying current) passes through the closed path? Since the principle of superposition applies to the magnetic field, the closed path line integral will just equal μ_0 times the sum of the currents that pass through the path. Whether the current is + or - is determined by the "right hand rule" described above: If the fingers of your right hand curl in the direction of the path, then your right thumb will point in the direction of positive current.

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 I_{Net\ through\ path} \tag{27}$$

Comments:

1. The little circle in the middle of the integral sign means that the path integral is closed. For the equation to apply, **the line integral must be over a closed path**.

2. In lecture we have only justified this equation for straight wires of infinite length. Using vector calculus, it can be shown that this result is valid for any wire configuration as long as the wires (or currents) do not have an open end. That is, if all the wires are closed loops or extent to infinity then the above equation is valid. A counter-example is a discharging parallel plate capacitor. In this case the current starts at one plate and ends at the other. There is a gap in the middle where there is no electrical current (i.e. no charge flow).

3. If the path does not enclose any current, then the closed line integral on the left is zero.

The equation above is refered to as Ampere's Law. In lecture we will do some classic examples using Ampere's law: The infinite solenoid, coaxial cable, and the toroid. In these examples our goal will be to determine the magnetic fields produced by these wire configurations. However, before we can uniquely calculate these magnetic fields, we need one more global property of magnetic fields: what is the surface integral over a closed surface for magnetic fields?

For the electrostatic field $\vec{E}$ we showed that Coulombs Law resulted in $\oint\oint \vec{E} \cdot d\vec{A} = Q_{enclosed}/\epsilon_0$. This was because electric fields are produced by a stationary source of charge that eminate from the source and whose magnitude decreases as $1/r^2$. If there are no stationary sources within the closed surface, then the surface integral equals zero. For the magnetic field, **there are no stationary sources**. That is, there are no magnetic monopoles. At least until now, none have been discovered. So until one is discovered, we must have

$$\oint\oint \vec{B} \cdot d\vec{A} = 0 \tag{28}$$

for **any closed surface**! Whereas $\oint \vec{B} \cdot \vec{dr}$ is only true when the current sources have no open ends, the surface integral is valid for any source of magnetic fields.

We can use the two integral properties of the magnetic field:

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 I_{Net\ through\ path}$$
$$\oint \oint \vec{B} \cdot d\vec{A} = 0$$

The first one is valid for current sources with no open ends, and the second is always true. Next I'll briefly present the results for the magnetic field inside an infinite solenoid and a toroid, with more details given in lecture.

Infinite Solenoid

We previously showed, using the Biot-Savart Law, that the magnetic field on the axis of an infinitely long solenoid was $B_x = \mu_0 In$, where n equals the number of turns per length. What about the magnetic field at points inside the solenoid but not on the axis? We will use the fact that the solenoid has axial symmetry about its central axis.

Consider a point located a distance r from the axis of an infinite solenoid. At this point, the magnetic field could have three different components: 1) parallel to the axis of the solenoid (B_x), 2) radially away from the axis (B_r), or 3) circular around the axis (B_θ). Because of axial symmetry, these components can only depend on r and not on the position along the axis (x) or the angle around the axis (θ). That is the components can only have an r dependence: $B_x(r)$, $B_r(r)$ and $B_\theta(r)$.

If we choose a closed cylindrical surface of radius r, length l whose axis coincides with the axis of the solenoid, then $\oint \oint \vec{B} \cdot d\vec{A} = 2\pi \mathrm{r} l B_r$. However, this surface integral must be zero. Therefore, $B_r = 0$. If we choose as a path, a circle of radius r whose axis is on the axis of the solenoid, then $\oint \vec{B} \cdot d\vec{r} = 2\pi r B_\theta$. Since there is no current flowing through the path, this line integral must be zero. Thus, B_θ must be zero. Finally, we choose as a path a rectangle of dimensions $l \times r$ whose side l lies on the axis of the solenoid. Then $\oint \vec{B} \cdot d\vec{r} = \mu_0 Inl - B_r l + 0 + 0$. Since there is no current flowing through the path this line integral must be zero. Thus, we have $l(\mu_0 In - B_r) = 0$ or $B_r = \mu_0 In$ in the x-direction. The magnetic field inside an infinite solenoid is constant and equal to μIn everywhere inside the solenoid. In lecture we will discuss the magnetic field outside the solenoid.

Toroid

To make a toroid, one could coil wire around a donut, then take the donut out. Lets consider the magnetic field inside a toroid that has a rectangular cross section.

Let the inner radius be a and the outer radius b, and the height of the cross section be h. Suppose the toroid has N total turns of wire. The toroid has axial symmetry about an axis through its center and perpendicular to the plane of the toroid.

Consider a point inside the toriod a distance r from the central axis. That is $a < r < b$. The magnetic field at this point could have three components: 1) radially away from the axis of symmetry (B_r), 2) circular around the axis of symmetry (B_θ), or 3) "up or down" in the $\pm h$ direction (B_h). We will show in lecture that because of the symmetry (axial and h-direction), the integral $\oint \oint \vec{B} \cdot \vec{dA} = 0$ leads to $B_r = 0$ and $B_h = 0$.

If we choose a circular path of radius r that goes around the axis of symmetry and is inside the toroid. Then, $\oint \vec{B} \cdot \vec{dr} = 2\pi r B_\theta$. From Ampere's law this closed path integral equals the current that passes through the path, which is NI. Thus, $B_\theta = \mu_0 NI/(2\pi r)$ for points inside the toroid. In lecture we will consider points outside the toroid.

Magnetic Force

Up to now we have discussed some sources for the magnetic field. The main type of source we have dealt with has been a wire that has a current flowing through it. As far as the magnetic force goes, we have only considered how a magnetic field affects small bar magnets. In lecture we used our magnaprobe to observe magnetic fields. Consider the following question: *will an object that has charge feel a magnetic force*?. That is, is there any other object besides a small bar magnet that experience a magnetic force. We will do two experiments in lecture:

Experiment 1. We will place a wire in the presence of a strong magnetic field. When there is **no current** in the wire, the wire experiences **no force**. When current flows in the wire, the wire jumps. We will see that the direction of the force on the wire is *perpendicular to both the direction of the magnetic field and the direction of the current.*

Experiment 2. We will bring into lecture an "electron gun". Electrons will be sent forward in a tube and will hit the screen in front of the tube. Electrons have a charge e. When a magnetic field is introduced in the path of the electrons, the electrons are deflected. We will see that the force on the electrons is *perpendicular to both the direction of the magnetic field and the direction of the electrons velocity.*

Since current flow is essentially a flow of charge, the same effect is happening in both experiments. If an object that has a net charge moves in the presence of a magnetic field, it feels a force. The force is perpendicular to both the magnetic field and the velocity of the object.

This might seem strange at first, since we are used to forces being in towards or away from a source, or along the direction of a field vector. The gravitational force is always towards the other object. The electrostatic force is always away or towards its source, or along the direction of $\vec{E}$: $F_{electrostatic} = q\vec{E}$. However, these two forces do not depend on the velocity of the object. The force is the same if the object is moving or not. The magnetic force, on the other hand, **depends on the velocity of the object**. If the object is not moving, there is no force (the wire felt no force if the charges in it had an average velocity equal to zero). Thus, the magnetic force vector must depend on two other vectors: the magnetic field $\vec{B}$ and the velocity vector $\vec{v}$ of the object. The only mathematical operation (in three dimensions) that takes two vectors and produces a third vector in a mathematically invarient way is the cross product. That is,

$$\vec{F}_{magnetic} \sim \vec{v} \times \vec{B} \tag{29}$$

The net charge also plays a role in the magnetic force on a particle. If a particle does not possess charge, it will not feel a magnetic force (or electric force). Electrons feel an opposite force that protons feel if they move with the same velocity in the same magnetic field. If an object has N_p protons and N_e electrons, then the superposition principle for forces yields:

$$\vec{F}_{magnetic} \sim (N_p - N_e)e\vec{v} \times \vec{B} \tag{30}$$

In terms of the net charge on the object, this equation becomes:

$$\vec{F}_{magnetic} \sim q\vec{v} \times \vec{B} \tag{31}$$

where q is the net (or total) charge on the object or particle. From this force equation, we can see that the units of magnetic field are Force per charge per velocity, or in more basic quantities: mass/(charge-sec). As with the electric field, the force equation can be used to define $\vec{B}$, and hence the $\sim$ can be replaced with an equal sign:

$$\vec{F}_{magnetic} = q\vec{v} \times \vec{B} \tag{32}$$

Before we discuss the force a wire feels in the presence of a magnetic field, the above equation deserves some comments:

1. The magnitude of the magnetic force is $|\vec{v}||\vec{B}|sin(\theta)$ where θ is the angle between the magnetic field and the particle's velocity. Thus, if the particle moves in the direction of the magnetic field, it does not feel a magnetic force. Only the component of the velocity perpendicular to the magnetic field contributes to the force.

2. Since the magnetic force is always perpendicular to the particles velocity, **the magnetic force cannot do any work** on the particle. The magnetic force cannot change the speed of a particle, it can only change the direction of its motion.

3. The magnetic field is similiar to the electric field in that is it a "force per charge". However, in the case of the magnetic field it is a (force per charge)/speed.

4. It is remarkable that both the electrostatic force and the magnetic force are proportional to the same quantity: charge. There is something to be learned from this property.

Magnetic Force on a Straight wire with Current

From the equation for the magnetic force on a particle, we can calculate the magnetic force on a piece of *straight wire* that has a current flowing through it. Suppose we have a wire that has n "charge carriers" per unit volume and each charge carrier has a charge q. The charge carriers are usually electrons. Suppose each of the charge carriers is moving down the wire with a drift velocity of $\vec{v}_d$. Let the wire have a length L, a cross sectional area of A, and a current I. If there is a magnetic field $\vec{B}$ at the location of the wire, then the net force on the straight wire is

$$\vec{F} = nALq(\vec{v}_d \times \vec{B}) \tag{33}$$

since n is the number of charge carries per volume. Note that $nALq$ equals the total charge that is moving. Remembering that the current density $\vec{J} = nq\vec{v}_d$ we have

$$\vec{F} = AL(\vec{J} \times \vec{B}) \tag{34}$$

This equation is in a nice form. Remember that the current I equals $A|\vec{J}|$, so we can replace $A\vec{J}$ with I. However, if we do this we need to specify the direction of the current. This is best done by expressing L as a vector: $\vec{L}$, where the vector $\vec{L}$ points in the direction of the current. With this convention, we have

$$\vec{F} = I(\vec{L} \times \vec{B}) \tag{35}$$

Comments

1. This equation is only valid for a *straight wire*, which has a current I. However, if we have a curved wire, we can break it up into small pieces which can be considered as straight wires in the limit that the "piece size" goes to zero. We need then to integrate over the wire.

2. The force on the wire is always perpendicular to the wire.

3. The force is the same whether the current is caused by positive charges moving one way or negative charges moving the other way.

In lecture we will do examples of the magnetic force on wires and individual particles.

Units of the Electromagnetic Interaction

Now that we understand the electric (velocity independent) force and the magnetic (velocity dependent) force we can discuss units. Consider two point particles, one with a charge q_1 and the other a charge of q_2 that are separated by a displacement $\vec{r}$. Suppose particle "1" has a velocity $\vec{v}_1$ and particle "2" a velocity of $\vec{v}_2$.

The electric force $\vec{F}_e$ between the two particles is

$$\vec{F}_e = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r} \tag{36}$$

from Coulomb's law.

To find the magnetic force between the two particles, it is best to start with the force between charges moving in a wire and a charge outside the wire. If charges are moving in a wire of length $\vec{\Delta l}$, then by the Biot-Savart Law, the magnetic field produced is

$$\Delta\vec{B} = \frac{\mu_0}{4\pi} I \frac{\vec{\Delta l} \times \hat{r}}{r^2} \tag{37}$$

The current I can be written as $I = |\vec{J}|A$. $\vec{J}$ is equal to $nq\vec{v}$. Since $nA\Delta l$ is equal to the total number of charge carriers N in the wire segment Δl. So the magnetic field produced by the line segment can be written as

$$\Delta\vec{B} = \frac{\mu_0}{4\pi} N q \frac{\vec{v} \times \hat{r}}{r^2} \tag{38}$$

From this equation, we can interpret the magnetic field produced by a single charge q_1 moving with a velocity $\vec{v_1}$ as

$$\Delta\vec{B} \approx \frac{\mu_0}{4\pi} q_1 \frac{\vec{v}_1 \times \hat{r}}{r^2} \tag{39}$$

I have put in the approximation sign, since there are small corrections to this equation due to relativity (Einstein's Special Relativity). If particle "2" is moving with a velocity of $\vec{v}_2$ at the location $\vec{r}$, then the magnetic force it feels is

$$\vec{F}_m = q_2(\vec{v}_2 \times \Delta\vec{B}) \tag{40}$$

using the expression above for $\Delta\vec{B}$ we have

$$\vec{F}_m \approx \frac{\mu_0}{4\pi} q_1 q_2 \frac{\vec{v}_2 \times \vec{v}_1 \times \hat{r}}{r^2} \tag{41}$$

Note: to have a magnetic force *both* charges have to be moving. Summarizing:

$$\begin{aligned} \vec{F}_e &= \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r} \\ \vec{F}_m &\approx \frac{\mu_0}{4\pi} q_1 q_2 \frac{\vec{v}_2 \times \vec{v}_1 \times \hat{r}}{r^2} \end{aligned}$$

These equations give us guidance in choosing the units for the electromagnetic interaction. From mechanics we have the basic units for length, time and mass. The unit of force is derived from these three fundamental units. We now need to consider units for charge, and corresponding values for ϵ_0 and μ_0. We have some options, and we list just two:

Option 1: We choose a standard amount of charge as our unit charge. Then by measuring the electrostatic force between two point charges, the first equation allows us to determine ϵ_0. By measuring the magnetic force between point charges that are moving, the second equation allows us to determine μ_0. One could choose e as the standard unit of charge. Then the proton would have a charge of $+1$ and the electron a charge of -1 in these units.

Option 2: We could choose our unit of charge so that one of the constants, μ_0 or ϵ_0, equals one or another appropriate number in these units. Then the other equation would give us a way to measure the other constant.

Most texts, and the metric system, use the second option. That is, the unit of charge is the Coulomb, and one Coulomb is that amount of charge that makes μ_0 exactly equal to $4\pi \times 10^{-7}$ in metric units. That is why you see in the textbooks that $\mu_0 = 4\pi \times 10^{-7}$ Tm/A. The electrostatic equation can be used to determine ϵ_0 which comes out to be $\epsilon_0 \approx 8.85 \times 10^{-12}$ $\mathrm{C}^2/(\mathrm{Nm}^2)$.

There are some very interesting properties of the two equations above:

1. **No matter what one chooses as the basic unit for charge, the product $\epsilon_0\mu_0$ is always the same**. This can be seen as follows. Let q_0 be the unit of charge that results in a value of ϵ_0 and μ_0. Suppose we change the unit of charge to $q = xq_0$, where x is a scale factor. The new ϵ_0 would increase by the factor x: $\epsilon_0' = x\epsilon_0$, and the new μ_0 would decrease by the factor x: $\mu_0' = \mu_0/x$. The new product of ϵ_0' times μ_0' would equal the old one $\epsilon_0\mu_0$ since the factor x would cancel. Note that the product $\epsilon_0\mu_0$ has units of s^2/m^2 or one over speed squared! There are no units of charge in the product.

2. **The combination $1/\sqrt{\epsilon_0\mu_0}$ has units of speed, which is independent of our choice of charge unit**. Thus, there is a fundamental speed associated with the electromagnetic interaction. In metric units, $1/\sqrt{\epsilon_0\mu_0} \approx 3 \times 10^8$ m/s, the speed of light. More on this later.

3. We can see the relative importance of the two forces for interacting particles by taking the ratio of the magnetic force to the electrostatic force. From the above equations we have:

$$\frac{F_m}{F_e} \sim \epsilon_0\mu_0 v_1 v_2 \tag{42}$$

where the $\sim$ sign is used since we have left out the affect of the relative angles between the velocities. Note that $\epsilon_0\mu_0 = 1/c^2$, where $c \equiv 3 \times 10^8$ m/s (the speed of light). Thus, for speeds much less than the speed of light the magnetic force between individual particles will be less than the electrostatic force between them.

4. The magnetic force $\vec{F}_m$ depends on the particle's velocities. However, velocity is relative to one's reference frame. This means that **a magnetic force for one observer might be an electric force for another**. The electric force is also reference frame dependent. These are important considerations in formulating the laws of electromagnetism in a way that have the same form for all inertial observers, and led Einstein to the theory of special relativity.

The effect of Changing Magnetic Fields

We have talked about the sources for electric and magnetic fields, and the forces that they cause on a charged particle. We have expressed the fields in terms of the charge sources using Coulomb's Law and the Biot-Savart Law:

$$\begin{aligned}
\vec{E} &= k\frac{Q}{r^2}\hat{r} \\
\Delta\vec{B} &= \frac{\mu_0}{4\pi}I\frac{\vec{\Delta l} \times \hat{r}}{r^2}
\end{aligned}$$

We also used the line and surface integrals, Gauss' Law and Ampere's Law, to express the properties of the fields:

$$\begin{aligned}
\oint\oint \vec{E} \cdot d\vec{A} &= \frac{Q_{net\ enclosed}}{\epsilon_0} \\
\oint\oint \vec{B} \cdot d\vec{A} &= 0 \\
\oint \vec{E} \cdot d\vec{r} &= 0 \\
\oint \vec{B} \cdot d\vec{r} &= \mu_0 I_{net\ through\ path}
\end{aligned}$$

You might think that there is nothing more to understand about the sources of electric and magnetic fields. However, there is something missing in the equations above. It is due to the velocity dependence of the magetic force. If one goes into the reference frame of a moving particle, there is no magnetic force. However, there must be some force to take its place. In this "moving" reference frame, the magnetic fields are moving. Therefore, a changing magnetic field must produce some sort of force that we haven't accounted for yet. We will demonstrate this phenomena with an example.

Suppose we have a bar magnet and a closed rectangular loop of conducting wire. Let the bar magnet be stationary at the origin and point in the +z direction. Now suppose the rectangular loop of wire lies in the x-y plane and moves with constant velocity v_0 in the +x direction, $\vec{v} = v_0\hat{i}$. Let the sides of the rectangular wire loop be a and b. Suppose at $t = 0$ one side of the wire (a) passes directly over the bar magnet. That is, the front end of the rectangle is at position $x = vt$. Suppose the magnet field is strong only near the origin. We will view the situations from two different reference frames: one in which the magnet is at rest and the wire is moving, and the other in which the wire is at rest and the magnet is moving.

Reference frame in which magnet is at rest

In this reference frame, the the free charges in the front side of the wire are moving in the presence of a magnetic field and will feel a force. However, since only the front side of length a is over the magnet, only these charges will feel a force. The net result is that overall the charges are forced around the wire and a current is produced. We will demonstrate this in class. We will produce a current in the closed wire loop without batteries! As far as the current is concerned, the key quantity is the "voltage" around the wire loop. The "voltage" is essentially the force/charge times Δl integrated around the wire loop. In our case, this integral becomes:

$$\oint_{wire} \frac{\vec{F}}{q} \cdot \vec{dr} = Bva \tag{43}$$

The force acts **only on the charges on the front side** of length a over the magnet. The force/charge is Bv and the distance it acts is a. So the force per charge times distance around the loop is Bva, which is essentially the voltage around the wire loop. If the wire has a resistance R, then the current is $I = Bva/R$.

Reference frame in which the wire is at rest

This reference frame is moving with velocity $\vec{v} = v_0\hat{i}$ with respect to the other frame. Here, the loop is at rest and the magnet is moving with velocity $-v_0\hat{i}$. In this frame the free charges in the wire loop are not moving, i.e. have zero average velocity. Thus, as far as this observer is concerned the charges do not feel a magnetic force. Also, there is no electrical force since there are no charge sources. So, if this observer uses the equations of electrodynamics that we have developed so far, the observer will not predict that a current is formed in the loop. When we do the experiment in class, you will see that keeping the loop fixed and moving the magnet also produces a current in the loop. Thus, there is something missing in the equations above. In this frame, the wire is at rest but the magnetic field is changing. Thus, somehow *a changing magnetic field produces some sort of electric field.*

To fix the problem, we need to go into the first reference frame, since we are correctly predicting the experimental outcome, and try and express $\oint_{wire} \vec{F} \cdot \vec{dl}$ without the velocity v but with only a changing magnetic field.

Reference frame in which magnet is at rest

Since the position of the front side of the wire is $x = vt$, we can express the velocity v as $v = \Delta x / \Delta t$. Substituting for the force/charge times distance around the loop:

$$\oint_{wire} \frac{\vec{F}}{q} \cdot \vec{dr} = B \frac{\Delta x}{\Delta t} a \tag{44}$$

However, in this reference frame, B and a are constant, so we can write the above equation as:

$$\oint_{wire} \frac{\vec{F}}{q} \cdot \vec{dr} = \frac{\Delta(xBa)}{\Delta t} \tag{45}$$

This is a better form, since there is no velocity in the expression. The changing is a combination of the magnetic field times x. In one frame B is changing and x is not, and in the other frame x is changing and B is not. The quantity xaB is the magnetic flux through the wire loop. We can restate the equation more succinctly as

$$\oint_{wire} \frac{\vec{F}}{q} \cdot \vec{dr} = -\frac{\Delta \Phi_B}{\Delta t} \tag{46}$$

Where Φ_B is the total magnetic flux through the wire loop. The minus sign refers to the direction of the current. When the limit $\Delta t \to 0$ is taken, the right side becomes the time derivative of Φ_B.

$$\oint_{wire} \frac{\vec{F}}{q} \cdot \vec{dr} = -\frac{d\Phi_B}{dt} \tag{47}$$

This is a form that both observers can agree on: the right side has the same value in both reference frames. In one frame the wire is moving (magnetic field not changing) and in the other the magnetic field is changing and the wire is not moving. It can be shown that the above equation is true any time there is a changing magnetic flux through a closed wire loop. The magnetic flux can change if

a) the wire moves in a stationary magnetic field
b) the wire is stationary and the magnetic field inside the wire loop changes.
c) the wire's area changes in a stationary magnetic field
d) the wire is moving in an area in which the magnetic field is changing.

The closed integral around the wire loop, $\oint_{wire} \vec{F}/q \cdot \vec{dl}$ is often called the *E.M.F.* or electromotive force. It represents the voltage around the wire loop that is caused by an electric field produced by a (changing) magnetic field. It is often given the symbol ξ:

$$\xi \equiv \oint_{wire} \frac{\vec{F}}{q} \cdot \vec{dr} = -\frac{d\Phi_B}{dt} \tag{48}$$

The above equation (or some form of it) is called Faraday's Law after the discoverer of this rather complicated phenomena. It was an important insight to realize that the easiest way to express the effects of a changing magnetic field is through line and surface integrals, i.e. magnetic flux. One reason we covered electric flux and Gauss's Law was was to introduce you to the concept of flux and prepare you for Faraday's law. Faraday's discovery was one of the most important contributions to science. It revolutionized the world, and we will demonstrate its significance with numberous examples in lecture.

Modifying the Field Equations

We are how in a position to change the field equations. As mentioned, the easiest way to include the effects of a changing magnetic field is through a line integral: $\oint \vec{F}/q \cdot \vec{dl}$. Thus, it is most convenient to use the integral formulation for the fields, and not the Coulomb or Biot-Savart laws. In the integral forms, **the line and surface integrals are done over stationary mathematical topologies**. Although we have no wires, we must use the result for stationary wires. If the wire for the closed loop is stationary, then $\vec{F}/q$ is only caused by an electric field: $\vec{F}/q = \vec{E}$. This gives

$$\oint \vec{E} \cdot \vec{dr} = -\frac{d\Phi_B}{dt} \tag{49}$$

where the subscript "wire" on the integral is removed. This equation is true for any closed mathematical path. The corrected equations can now be written as:

$$\begin{aligned}
\oint\oint \vec{E}\cdot d\vec{A} &= \frac{Q_{net\ enclosed}}{\epsilon_0} \\
\oint\oint \vec{B}\cdot d\vec{A} &= 0 \\
\oint \vec{E}\cdot d\vec{r} &= -\frac{d\Phi_B}{dt} \\
\oint \vec{B}\cdot d\vec{r} &= \mu_0 I_{net\ through\ path}
\end{aligned}$$

It is important to note that the path and surface integrals on the left sides of the equal signs can only be evaluated over paths and surfaces that are not moving. The new addition takes into account the effect of changing magnetic fields. Indirectly it states that a changing magnetic field **induces** an electric field. This phenomena is often called induction. It is an extremely important addition to the laws of electrodynamics and deserves some comments.

1. The time derivative of the magnetic flux, $d\Phi_B/dt$ in the above equation is often written explicitly as an integral: $\Phi_B = \oint\oint \vec{B}\cdot d\vec{A}$:

$$\oint \vec{E}\cdot d\vec{r} = -\frac{d}{dt}\oint\oint \vec{B}\cdot d\vec{A} \tag{50}$$

The direction of the path for the path integral on the left side defines the direction for the area vector on the right. If the fingers of your right hand curl in the direction of the path, then your right thumb points in the direction of positive area.

2. The minus sign in the induction equation signifies the direction of the induced electric field or current. However, it is often easier to use another method stated by Lenz. Lenz's Law states that *the magnetic field produced by the induced current opposes the change in the magnetic field causing the induced current.*

We conclude the topic of magnetic induction with some examples.

A coil of wire rotating in a constant magnetic field

Suppose there is a constant magnetic field of magnitude B. Suppose we rotate a closed loop of wire, of area A, with the axis of rotation perpendicular to the magnetic field. Let the angle between the area vector and the magnetic field be θ. When $\theta = 0$ then the magnetic flux through the wire loop is maximized since $\vec{B}$ is perpendicular to the area: $\Phi_m = BA$. When $\theta = 90°$ then the magnetic flux through the loop is zero. For arbitary angle, $\Phi_m = BAcos\theta$. Note, in this case the magnetic flux is just (magnetic field) times (area) **since the magnetic field is constant inside the wire loop**. If the loop is rotating with a constant angular velocity ω, then $\theta = \omega t$:

$$\Phi_m = BAcos(\omega t) \tag{51}$$

The magnitude of the EMF generated in the wire loop is found by taking the time derivative of Φ_m:

$$\begin{aligned} |E.M.F.| &= |\frac{d\Phi}{dt}| \\ &= \omega BAsin(\omega t) \end{aligned}$$

A wire rotating at a uniform rate in a constant magnetic field will produce a sinusoidal "voltage" or current in the loop! This is a very convenient way to produce electrical current from mechanical motion. We refer to this type of device as an **electric generator**. The only peculiarity is that the current produced is sinusoidal in time, that is, it is alternating current (AC). This is one reason why we have alternating current delivered to our homes: it is easy to produce it by mechanical means.

Wire moving in a non-uniform magnetic field

Suppose there is an infinitely long wire with a steady current I flowing through it. Let there be another wire which is a rectangular with sides a and b. Let side "b" be parallel to the long wire and located a distance x from it. Let side a point directly to the long wire. Lets first calculate the magnetic flux **through** the rectangular wire **due to** the magnetic field produced by the infinitely long wire.

The magnetic field produced from the long wire has a magnitude of $\mu_0 I/(2\pi r)$, where r is the perpendicular distance from the wire. The magnetic field is perpendicular to the closed rectangular wire. Since the magnetic field is **not constant** inside the wire loop, the flux is **not equal** to just (Magnetic Field) times (area). We need to integrate over the loop area. We divide the area into strips of length b and width Δr, where the strip is parallel to the long wire. Then the flux through a strip is:

$$\Delta\Phi_m = \frac{\mu_0 I}{2\pi r} b\Delta r \tag{52}$$

Adding up all strips and taking the limit as $\Delta r \to 0$ yields an integral:

$$\Phi_m = \int_x^{x+a} \frac{\mu_0 I b}{2\pi r} dr \tag{53}$$

This integral is easily evaluated to be

$$\Phi_m = \frac{\mu_0 I b}{2\pi}(ln(a+x) - ln(x)) \tag{54}$$

If I remains constant and the rectangular wire does not move, then no current is induced in it. This is because the magnetic flux Φ_m is not changing. To change the flux through the wire we need to either move the one of the wires, or change the current in the long wire. Suppose we move the rectanglar wire away with a constant speed $v = dx/dt$. To find the change in the flux with time, we need to differentiate the above equation with respect to time:

$$\frac{d\Phi_m}{dt} = \frac{d\Phi_m}{dx}\frac{dx}{dt} \tag{55}$$

Since $v = dx/dt$ we obtain

$$|\frac{d\Phi_m}{dt}| = \frac{\mu_0 I b v}{2\pi} \frac{a}{(a+x)x} \tag{56}$$

which is the $E.M.F.$ generated in the rectangular wire loop.

Another way to generate a current in the rectangular loop is to keep it stationary and change the current in the long wire. In this case, we have

$$E.M.F. = \frac{d\Phi_m}{dt} = \frac{\mu_0 b}{2\pi}(ln(a+x) - ln(x)) \frac{dI}{dt} \tag{57}$$

This is an interesting way to generate current in the rectangular wire. Neither wire needs to be moving! **A changing current in one wire produces a current in the other**. Next we will do another example of (changing current in one wire) $\to$ current in another wire.

Changing Current producing Changing Current in stationary wires

Suppose we have a solenoid of length l that has a total of N_1 coils of wire. Let the area of the solenoid be A. Around the solenoid we place a circular coil of wire that has N_2 loops of wire. Suppose the current in the solenoid is I_1. What is the magnetic flux through the coil of wire that is around the solenoid?

This is a simple calculation if we approximate the solenoid (wire 1) to be infinite: $B \approx \mu_0 n I_1$ where $n = N_1/l$, the number of turns per length. Since the magnetic field only occupies an area A (of the outer coils of wire), we have

$$\Phi_2 \approx \frac{\mu_0 N_1 I_1}{l} N_2 A \tag{58}$$

where Φ_2 is the magnetic flux though the outer coil of wire (wire 2). If I_1 changes in time, then the magnetic flux through the outer coil will change and an E.M.F. will be produced in this coil:

$$E.M.F._2 = \frac{d\Phi_2}{dt} = \frac{\mu_0 N_1 N_2 A}{l} \frac{dI_1}{dt} \tag{59}$$

Suppose there is a sinusoidal current in the solenoid (wire 1): $I_1 = I_0 sin(\omega t)$. Then, a sinusoidal current will be generated in the other wire!!:

$$E.M.F._2 = \frac{\mu_0 N_1 N_2 A}{l} \omega I_0 cos(\omega t) \tag{60}$$

This is great. A sinusoidal current in one wire produces a sinusoidal current in the other. The nice thing is that by choosing appropriate values for N_1 and N_2 we can control the $E.M.F.$ generated in wire "2" (the outer coil). We can boost the voltage up or down depending on the number of turns we have. This sort of device is called a **tranformer**. Note that the transformer only works with **changing current**. It will not work with direct current D.C. This is another reason why the electrical voltage that comes to our homes is alternating (A.C.). It is easy to generate, and it is easy to change (or transform) to a higher or lower voltage!! In the U.S., the frequency of the sinusoidal voltage is 60 cycles/sec.

Magnetic Induction in Circuits

The results of the last two examples will apply to any two closed circuits: a changing current in one circuit will produce a current in another circuit. Let the two circuits be labeled "1" and "2". If circuit "1" has a current I_1, then a magnetic field $\vec{B}_1$ will be produced. This magnetic field will exist at the location of circuit "2". There will, therefore, be a magnetic flux through circuit "2" due to the magnetic field produced by circuit "1". We label this magnetic flux as Φ_{12}.

Since the magnetic field $\vec{B}_1$ is proportional to the current I_1 in circuit "1", the magnetic flux through circuit "2" will also be proportional to I_1:

$$\Phi_{12} \propto I_1 \tag{61}$$

We can replace the proportional sign with an equal sign and a constant:

$$\Phi_{12} = M_{12} I_1 \tag{62}$$

where M_{12} is called the **mutual inductance** of the two wire configurations.

You are probably thinking that a current in wire "2" will produce a magnetic flux through wire "1". In a similar manner, the magnetic flux through wire "1" will be proportional to the current in wire "2":

$$\Phi_{21} = M_{21} I_2 \tag{63}$$

where Φ_{21} is the flux through wire "1" due to the current in wire "2". In our upper division electrodynamics class we show that the mutual inductances are equal: $M_{12} = M_{21}$.

In the second example above, the mutual inductance between the infinite straight wire and the rectangular loop is $M_{12} = (\mu_0 b/(2\pi))\ ln((a + x)/x)$. For the solenoid and the outer coil, the nutual inductance is

$$M_{12} = \frac{\mu_0 N_1 N_2 A}{l} \tag{64}$$

Note that:

1. The mutual inductance only depends on the geometry of the two wires. It does not depend on the current flowing in the wires. There is no I in the equation for M_{12}. The mutual inductance is simple the *ratio* of the magnetic flux through the surface of one closed wire due to the magnetic field caused by a current in the other wire.

2. To have "mutual inductance" one needs two wires, and both should really be closed circuits. However, one wire could be an infinitely straight wire (which carries the current) and the other one closed.

3. Every two closed wire loops will have a mutual inductance, and electrical engineers will always need to consider this property of circuits.

Self Inductance:

If a wire carries a current, a magnetic field is produced. If the wire is closed, then this magnetic field will produce a magnetic flux through the closed wire itself. Thus, if a closed wire carries a current, there is a magnetic flux Φ_m through the wire circuit that is caused by the current in the wire itself. If the current is doubled, then the magnetic field everywhere is doubled and so is the flux through the circuit:

$$\Phi_m \propto I \tag{65}$$

where I is the current in the circuit. The proportional sign can be replaced by an equal sign and a constant:

$$\Phi_m = LI \tag{66}$$

where the constant L is called the **self inductance** of the wire configuration (or circuit). Perhaps this property will become more clear with an example. Lets find the self-inductance of a solenoid.

Note: The self-inductance of a closed circuit depends only on the geometry of the wire. It does not depend on how much current flows in the wire. In fact, it is the ratio of the magnetic flux through the area of the wire loop caused by the current in the wire divided by the current that is in the wire.

Self-inductance of a solenoid

Suppose we have a solenoid that has a total of N coils (or turns) of wire. Let the solenoid have a cross sectional area A and a length l. To calculate the self-inductance, we first let the circuit have a current I. Then we calculate the magnetic field produced by this current. Finally we calculate the magnetic flux through the area of the circuit.

If we make the approximation that the magnetic field inside the soleniod is approximately the same as an infinitely long solenoid, we have

$$B \approx \mu_0 (\frac{N}{l}) I \tag{67}$$

For an infinite solenoid, the magnetic field is constant throughout the whole inside of the coil. The magnetic flux therefore through one loop of solenoid wire is $\Phi_m \approx AB \approx A\mu_0 NI/l$. Since there are N coils of wire, the total magnetic flux through the solenoid is:

$$\Phi_m \approx \frac{\mu_0 N^2 A}{l} \; I \tag{68}$$

From this equation, we see that the magnetic flux through the circuit is proportional to the current in the circuit. The constant of proportionality is the self-inductance L of the circuit:

$$L \approx \frac{\mu_0 N^2 A}{l} \tag{69}$$

It is interesting to note that:

a) As with mutual inductance, self-inductance only depends on the geometry of the closed wire circuit. It does not depend on the current I flowing in the wire.

b) Self-inductance has units of μ_0 times distance.

c) There is an N^2 dependence for L. The more turns of wire, the stronger the magnetic field is inside the solenoid. The more turns of wire, the more magnetic flux. This results in a N^2 factor for L.

Self-inductance is a very important property of a circuit. In some circuits inductors are placed in circuits for particular applications. Previously, we discussed Kirchoff's law for the sum of the voltage drops around a closed wire loop, and found the sum to be zero. This is correct if the current is not changing in the circuit. If the current is changing, then path integral of $\oint \vec{E} \cdot \vec{dl}$ is not zero, but rather $-d\Phi_m/dt$. Thus, the Kirchoff Law involving voltage changes around a closed loop must be modified:

$$\sum (Voltage\ drops) = -\oint \vec{E} \cdot \vec{dl} = \frac{d\Phi_m}{dt} \tag{70}$$

where the sum over voltages drops means around a closed path in the circuit. The direction of positive flux is determined by the right hand rule and the direction of the path taken to calculate the voltage drops. That is, if the fingers of your right hand curl in the path direction, then your thumb points in the direction of positive area for the magnetic flux.

Lets consider a simple example. Suppose we have a circuit that consists of a battery of voltage V and resistor of resistance R. Suppose the circuit has a self-inductance of L. The self-inductance could be caused by an inductor placed in the circuit, or just the self-inductance of the wires themselves. If we apply the modified Kirchoff Law to the circuit we have

$$V - IR = \frac{d\Phi_m}{dt} = L\frac{dI}{dt} \tag{71}$$

If we rearrainge the terms the equation can be viewed as

$$V - L\frac{dI}{dt} = IR \tag{72}$$

In this form, we see that the induction term $L(dI/dt)$ tends to decrease the voltage in the current loop. Electrical engineers sometimes call this term a "back E.M.F.". The self-induction (or inductive reactance) in the circuit tends to impede any change in the current.

In the original form:

$$V - IR = L\frac{dI}{dt} \tag{73}$$

the equation can be readily solved. By "cross-multiplying" we have

$$\int \frac{dt}{L} = \int \frac{dI}{V - IR} \tag{74}$$

If the switch is closed at $t = 0$, that is $I(0) = 0$, we have

$$\frac{t}{L} = \int_0^I \frac{dI'}{V - I'R} \tag{75}$$

whose solution is

$$I = \frac{V}{R}(1 - e^{-tR/L}) \tag{76}$$

Some comments on this result for the "L-R" circuit:

1. As $t \to \infty$ the current approaches V/R, which is the result without any self-inductance.

2. The "time constant" L/R is called the LR time constant. It is the time it takes for the current to reach $(1 - e^{-1})$ (63%) of its final value.

L-C Resonance Circuit

An important application is a circuit with an inductor and a capacitor in the same circuit. We will consider here only a simple circuit with a capacitor and inductor in series. Let the capacitor have a capacitance of C, and the inductor a self-inductance of L. Let $\pm Q$ be the charge on the plates of the capacitor. Applying Kirchoff's modified voltage equation around the circuit gives:

$$V_c = \frac{d\Phi_m}{dt} \tag{77}$$

where V_c is the voltage across the capacitor. If $\pm Q$ are the charges on each plate of the capacitor, $V_c = Q/C$. If the self-inductance of the inductor (i.e. the circuit) is L, we have

$$\frac{Q}{C} = L\frac{dI}{dt} \tag{78}$$

The current is the rate of change of the charge on the capacitor. If I flows from the positive plate of the capacitor, then $I = -(dQ)/(dt)$. Differentiating both sides with respect to t:

$$\frac{1}{C}\frac{dQ}{dt} = L\frac{d^2I}{dt^2} \tag{79}$$

Substituting in for $(dQ)/(dt)$ gives

$$-\frac{I}{C} = L\frac{d^2I}{dt^2} \tag{80}$$

Rearrainging terms yields the following differential equation:

$$\frac{d^2I}{dt^2} = -\frac{I}{LC} \tag{81}$$

The solution $I(t)$ is a function whose second derivative is minus itself. Functions with this property are sinusoidal functions: $I(t) = Asin(\omega t)$. Since $(d^2I)/(dt^2) = -\omega^2 Asin(\omega t)$ we see that

$$\omega = \frac{1}{\sqrt{LC}} \tag{82}$$

This is an interesting result. The current in the L-C circuit oscillates with a frequency of $f = \omega/(2\pi) = 1/(2\pi\sqrt{LC})$. This frequency is the resonance frequency of the circuit. If an inductor and capacitor are in series in an antenna, signals at the resonant frequency have a large amplitude. By varying C one can tune into ones favorite radio or TV station.

There are many applications of inductors in circuits: producing sparks in car motors, etc, and we will discuss some of these in lecture. Lets close this section on the magnetic interaction by summarizing the main points.

Summary

The nature of the magnetic interaction, is quite complicated. This is because both the source and force both depend on the motion of charged particles. The ideas we have discussed the past 4 weeks can be broken down to 3 main features: sources of magnetic fields, the magnetic force on a particle moving in a magnetic field, and the effect of changing magnetic fields.

Sources of Magnetic Fields

Moving charges produce magnetic fields. The magnetic field produced by a small piece of wire segment that carries a current I is described by the Biot-Savart law: $\Delta\vec{B} = ((\mu_0 I)/(4\pi))(\vec{\Delta l} \times \hat{r})/r^2$. Ampere's Law deals with the closed path integral of $\vec{B} \cdot \vec{dl}$: $\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{through\ path}$.

Magnetic Force

Particles with charge that are moving in a magnetic field feel a force. The force the particle feels is perpendicular to both the velocity vector $\vec{v}$ and the magnetic field $\vec{B}$: $F_m = q\vec{v} \times \vec{B}$. Applying this force law to the charges moving in a straight wire gives: $F_m = I(\vec{l} \times \vec{B})$, where $\vec{l}$ points along the wire segment in the direction of the current and whose magnitude equals the length of the wire.

Changing Magnetic Fields

Changing magnetic fields produce electric fields. The magnetic interaction depends on the velocity of both the source and the particle experiencing the force. Since velocity is a *change* in position, a changing magnetic (or electric) field should cause some sort of effect. Faraday formulated this effect for changing magnetic fields most succinctly:

$$\oint_{wire} \frac{\vec{F}}{q} \cdot \vec{dl} = -\frac{d\Phi_m}{dt} \tag{83}$$

Or in words: A changing magnetic flux through a wire loop produces a "voltage" around the wire loop.

Final Thoughts

There is still one piece missing in the complete picture of the classical electromagnetic interaction. If you continue with Phy234 you will cover the contribution of Maxwell. He realized that the equations we have so far are inconsistent with charge conservation, and that a changing electric field must produce some sort of magnetic interaction. We state here the complete set of source equations (in integral form) for the electric and magnetic fields:

$$\begin{aligned}
\oint \oint \vec{E} \cdot \vec{dA} &= \frac{Q_{net}}{\epsilon_0} \quad (Coulomb) \\
\oint \oint \vec{B} \cdot \vec{dA} &= 0 \\
\oint \vec{E} \cdot \vec{dr} &= -\frac{d\Phi_B}{dt} \quad (Faraday) \\
\oint \vec{B} \cdot \vec{dr} &= \mu_0 I_{net} + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt} \quad (Ampere + Maxwell)
\end{aligned}$$

These equations are refered to as "Maxwell's Equations" for the classical electrodynamic fields. The force equation $F = q\vec{E} + q(\vec{v} \times \vec{B})$ relates the force to the fields. These ideas were developed between 1800 and 1864, and led to a radical change in the lives of humans. The electric generator and transformer were developed using Faraday's law, which allowed engineers to bring energy into our homes. People could cook food in their kitchens from snow melting high in the mountains. Electric motors turned because of the physics contained in these equations. Maxwell's contribution connected the electric-magnetic interaction with light (or more generally electromagnetic radiation). Radio and television were made possible from an understanding of these equations. The discovery of the laws of classical electrodynamics was one of the most significant achievements of humankind in the 1800's.

The interaction of atoms and molecules with each other are almost entirely electric and magnetic. Thus, all chemical and biological properties derive at their most basic level from the laws of electrodynamics. When you raise your arm, digest your food, or use gasoline to move your car, the electromagnetic force is at work. Basically, thinking is electromagnetic in nature. So when you take your final exam on the laws of electricity and magnetism, you will be using these same laws to solve the questions. As far as I know, the electomagnetic interaction is the only one which can be used to understand itself. Think about that!