## Notes on Electrostatics

These notes are meant for my PHY133 lecture class, but all are free to use them and I hope they help. The ideas are presented roughly in the order in which they are taught in my class, and are designed to supplement the text. I will be writing these notes as I teach the class, so they will be constantly updated and modified.

### General features of the Electrostatic interaction

We will be begin our discussion by doing a simple demonstration in lecture. We start with a PVC pipe and a paper towel. Initially there is no force between the two. However, after rubbing the paper on the PVC pipe, we find they are attracted to each other! If we rub the paper towel on two similar PVC pipes, we now find that the two PVC pipes push each other apart (repulsion)! What kind of force is this, and what property of nature causes the force? The force is certainly not gravity, and the source is not the objects mass as it is in Newtonian gravity.

We call this the electrostatic (or electrical) interaction, and we call the property of nature that causes the force **charge**. It is quite different than the gravitational force, since the electrostatic force can be attractive or repulsive (whereas gravity is only attractive). Note that repulsion occured between the two similar PVC pipes. Whatever charge they had, they had the same charge. Thus, the experiments would imply that like charges repel.

What about attraction? Initially the PVC pipe and the paper did not attract or repel. However, after they were rubbed together then they attracted each other. We know that the PVC pipe acquired some charge, since two PVC pipes repel each other. The experiments would suggest that initially the paper and pipe each had zero **net** charge, and after rubbing, some charge was transfered from the paper to the pipe. If the paper was initially neutral (0 net charge), then the paper might have negative (or the opposite) charge as the pipe. Thus, "opposite" charges attract.

To understand what is really going on, we need to understand what the materials are made of. Today, we know that matter is made up of atoms. The atoms themselves are comprised of a massive nucleus (protons and neutrons) surrounded by electrons. When two **different** materials are rubbed together, some electrons can be rubbed off one and onto the other. The electrons and protons possess charge, as well as mass and other properties. However, it is the property of charge that produces the electrical force.

Consider the following set of experiments. First we place two electrons a distance of $r_0$ away from each other and measure the electrical force they exert on each other. We find that they *repel* each other with a force $F_0$. We now repeat the same experiment except we do it with protons separated by a distance of $r_0$. We find they also *repel* each other with the same force $F_0$. Finally, we place an electron and a proton a distance $r_0$ apart and measure the force they exert on each other. We find that they *attract* each other with a force of the same magnitude $F_0$. These experiments tell us that the electron and the proton have the same magnitude of charge, but are "opposite". We usually represent the amount of charge on the proton and the electron as $e$. Protons have been assigned a positive charge ($+e$), and hence electrons have a negative charge ($-e$). Neutrons have zero net charge. There is no electrostatic force between a neutron and either an electron or a proton. A neutral atom has an equal number of protons and electrons.

In the case of the PVC pipe and the paper, electrons were rubbed off of the paper onto the PVC pipe. The PVC pipe acquired an excess of electrons (more electrons than protons) and had an overall negative charge. The paper lost some of its electrons and had an overall positve charge. The extra electrons will not stay on the pipe forever, but will gradually be pulled off by moisture in the air. Thus, after a while both the pipe and the paper will be neutral again. On dry days, the objects can keep their charge for quite a while. Note: if two identical materials are rubbed together, charge is **not** transfered from one to the other.

### Quantitative properties of the Electrostatic Interaction

The last two quarters we have seen that many aspects of nature can be understood in "simple" mathematical terms, and we hope that this is true for the electrostatic interaction. As was done with the force of gravity, it is best to start with the simplest objects and build up to more complicated ones. Let's consider the electrical force between two very small "point" particles that possess charge. Let particle "1" have an amount of charge equal to $q_1$, and let particle "2" have an amount of charge equal to $q_2$. Suppose they are separated by a distance $r$ and are not moving. How does the electrostatic force depend on $q_1$, $q_2$, and $r$?

In lecture we will demonstrate that if the number of electrons transfered to the PVC pipes increases, so does the force between the pipes. We will also show that the force gets stronger as $r$ decreases.

### Units for Charge

In past quarters we have discussed the units of the fundamental quantities of mass, length, and time, as well as derived quantities such as velocity, acceleration and force. For the electrostatic force we have this new quantity: Charge. Physicists believe that charge is a fundamental quantity, and will have its own unit. However, there is one interesting property of charge that mass, length and time do not have:

charge is quantized in multiples of a basic charge. This means that the charge on any particle or object is an integer multiple of a fundamental charge unit called $e$. For the three particles that make up all of our stable matter the chart below lists their charges and masses:

| Particle | Charge | Mass (Kg) |
|---|---|---|
| proton | $+e$ | $1.673 \times 10^{-27}$ |
| neutron | 0 | $1.675 \times 10^{-27}$ |
| electron | $-e$ | $9.11 \times 10^{-31}$ |

Since all stable matter is made up of protons, neutrons, and electrons, the net charge $q$ of a material is $(N_p - N_e)e$ where $N_p$ is the number of protons in the material and $N_e$ is the number of electrons. If there are an equal number of protons and electrons, the material is neutral. If the material has an excess of electrons (protons), then the material has a net negative (positive) charge. Thus, $q = (N_p - N_e)e$ is quantized in multiples of $e$ since $N_p$ and $N_e$ are integers. This quantization doesn't only hold for stable mater, but for all subatomic particles that are directly observed in the laboratory (quarks having a charge that is a fraction of $e$).

Thus, one really doesn't need a special unit to denote the charge on an object. One only needs an integer denoting the number of fundamental charges on the object. Whether $e$ itself has units is a question that is considered in quantum mechanics. We will take the classical physics approach and assign a unit to charge, that being the **Coulomb**. We will return to the discussion of charge units when we talk about the magnetic force. In terms of Coulombs, $e \approx 1.6 \times 10^{-19}$ Coulombs. The charge on any material object is equal to: $q = (N_p - N_e)1.6 \times 10^{-19}$ Coulombs, and is consequently quantized.

When macroscopic objects are "charged" large numbers of electrons are transfered (around $10^{10}$ or more). Because of this, we are not sensive to the quantization of charge and tend to think of charge as a continuous quantity. We might say that an object has a charge of 2 Coulombs, which might not be an exact multiple of $e$. As we go through the course, try not to forget that charge is quantized. Millikan (1909) was the first to demonstrate experimentally that charge is quantized, and received a Nobel Prize for his discovery.

### Electrostatic Force Law

Coulomb (1736-1806) carried out accurate experiments to determine how the electrostatic force depends on the distance between charged "point" objects as well on the magnitudes of their charges. He demonstrated that the magnitude of the electrostatic force varies as $1/r^2$ where $r$ is the distance between the point objects.

We showed in lecture that the electrostatic force increases when the objects have acquired more charge. Experiments show that the force is proportional to the product of the charges $q_1$ and $q_2$, where $q_i$ is the net charge on object $i$. This result follows from the superposition principle for the electrostatic force, and can be understood as follows: Suppose the magnitude of the force between two electrons separated a distance $r$ is $F_0$. If we replace one of the electrons with two electrons glued together, then the force between these two electrons and the other one will be $2F_0$, since we believe that electrostatic forces add like vectors. We can continue with this reasoning for any number of extra electrons. Suppose "point" object 1 has $n_1$ extra electrons, and "point" object 2 has $n_2$ extra electrons. Each electron on object 1 will feel a force of $n_2F_0$. Now, since there are $n_1$ electrons on object 1, the magnitude of the net force between them will be $n_1n_2F_0$:

$$|\vec{F}_{electrostatic}| \propto \frac{n_1 n_2}{r^2} \tag{1}$$

It is more convenient to express the electrostatic force law in terms of the net charge on the "point" objects instead of the numbers of excess (electron or protons). The net charge on object 1 is $q_1 = (N_e(1) - N_p(1))e$ and on object 2 $q_2 = (N_e(2) - N_p(2))e$, where $N_e(i)$ ($N_p(i)$) are the number of electrons (protons) on object "i". In terms of the net charges on the point object we have the usual form of Coulomb's Law:

$$|\vec{F}_{electrostatic}| \propto \frac{|q_1||q_2|}{r^2} \tag{2}$$

The absolute values are used since the charges can be positive or negative and the equation is an expression for the magnitude of the electrostatic force. **The direction of the force is along the line joining the two point particles**. If they are the same charge, then each particle feels a force directed away from the other one. If they charges are opposite, then each particle feels a force towards the other. The magnitudes are equal (Newton's Third Law). The proportionality sign can be replaced with an equal sign plus constant:

$$|\vec{F}_{electrostatic}| = k\frac{|q_1||q_2|}{r^2} \tag{3}$$

where $k \approx 9 \times 10^9\ Nm^2/C^2$ and carries the units of charge. The constant $k$ is also written in terms of the permittivity of free space $\epsilon_0$: $k = 1/(4\pi\epsilon_0)$, where $\epsilon_0 = 8.854 \times 10^{-12}\ C^2/(Nm^2)$.

It is sometimes convenient to express Coulomb's electrostatic force law with direction, that is to use vectors for the approriate quantites. If we define $\vec{F}_{12}$ as the force **on** point particle 2 **due to** point particle 1, then

$$\vec{F}_{12} = k\frac{q_1 q_2}{r^2}\hat{r}_{12} \quad (4)$$

where $\hat{r}_{12}$ is a unit vector pointing **from** particle 1 **to** particle 2. Note, that for like charges the product $q_1 q_2$ is positive and particle 2 feels a force away from particle 1, $+\hat{r}_{12}$. For unlike charges the product $q_1 q_2$ is negative the force 1 feels is toward 2, $-\hat{r}_{12}$.

**Comments on Coulomb's Law**

1. **The formula is only valid for point particles**. How small is small enough to be considered a "point particle"? If the largest dimension of the particles is $d$, and the separation of the particles is $r$, then if $d/r << 1$ the particles can be considered as points and the formula should apply.

2. **The mass of the particles does not effect the electrical force**. An objects mass will affect its motion, since mass is a measure of inertia. Mass and charge are very different. Mass has a dual role: it is a measure of resistance to change in motion, and it is the source of gravity (Newton). Charge does not add to the inertia of an object, nor does it affect the gravitational force.

3. **Why is the force called electro-static**? It is because the force does not depend on the velocity of the particles. The Coulomb's law force is the same if the particles are stationary or moving. We will see (later in the course) that if the charged particles are moving there is another force that depends on their speeds: the magnetic force. At first we will just consider the electrostatic part of the force.

4. **Why does the magnitude of the force decrease as** $1/r^2$. We have seen this inverse-square law dependence with the gravitational force law (Newton), as well as sound and light intensity. For light and sound intensity, the inverse-square law was a result of the energy being equally distributed on a spherical surface. Since the area of a sphere is $4\pi r^2$, the sound or light intensity decreased as $1/r^2$. For the electostatic force there is nothing that is spread out evenly over a spherical surface, however the $1/r^2$ dependence is a result of the geometry of space. A more advanced understanding of this dependence comes from quantum electrodynamics. Here we will marvel

over the simple inverse-square decrease of the electrical force and use this nice property to formulate Coulomb's law using geometry via Gauss' Law to be covered shortly.

5. Coulomb's Law holds at atomic length scales. It is one of the more important forces in the atom.

6. Between particles, the electrostatic force is much larger compared to the gravitational force. A stationary proton and a stationary electron are attracted to each other by two forces: gravity and the electrostatic force. They are attracted via gravity since they both have mass, and they are attracted via the electrostatic force since they both have charge. However, the electrical forces within the atom are much much stronger than the gravitational ones, which are usually neglected in calculations.

7. It is a bit mysterious that objects with charge can influence each other even though they are not touching. This so-called "action at a distance" is not very satisfying. Our present understanding, from quantum mechanics, is that the interaction can be described locally. The charge is related to the probability that a photon will be emitted or absorbed locally. If you want to know more about quantum electrodynamics you should be a physics major.

**Superposition Principle**

Coulomb's law holds for small "point" objects with charge, or small point charges. What if the objects are not small? The electrostatic force behaves like a normal force, and is a vector. That means that if point object "1" experiences an electrostatic force from two other point objects, say point object "2" and from point object "3", the net force on object "1" is the **vector sum** of the two forces:

$$\vec{F}_1(Net) = \vec{F}_{21} + \vec{F}_{31} \tag{5}$$

Remember that forces are vectors, so the sum on the right is a sum of **vectors**. We saw this same principle with the gravitational force, and it also holds for the electrostatic force. **The net electrostatic force on a point particle is the vector sum of the electrostatic forces due to all other point particles.** Stated in a more precise mathematical way:

$$\vec{F}_1(Net) = \sum_{i=2}^{\infty} k \frac{q_1 q_i}{r_{1i}^2} \hat{r}_{i1} \tag{6}$$

where $r_{1i}$ is the distance between point particle "1" and the i'th point particle, and the sum goes over all particles in the system. If there is a continuous (solid) distribution of charges, one can break up the solid into small "points" and sum **or integrate** over the solid. We will do a number of examples of the force on a point object with charge due to other point charges as well as continuous (or solid) objects.

Coulomb's law plus the superposition principle contains all the "physics" of the electrostatic force. For the next three weeks we will not be introducing any new principles of physics, but rather build upon this one important law of nature.

## Electric Field

A nice feature of the electrostatic force law is that **the force that a point particle feels is proportional to the charge the particle has**. Suppose at a particular point in space a particle with a charge of 1 Coulomb feels a force of 10 Newtons. If this particle were replaced with another point particle that had a charge of 2 Coulombs, the new particle would feel a force of 20 Newtons in the same direction as the first. A charge of 3 Coulombs $\rightarrow$ 30 Newtons, etc. This proportionality is evident from the equation above for the net force on particle "1" with charge $q_1$. The quantity $q_1$ factors out of the sum:

$$\vec{F}_1(Net) = q_1 \sum_{i=2}^{\infty} k \frac{q_i}{r_{1i}^2} \hat{r}_{i1} \tag{7}$$

The force on a particle located at the point where particle "1" is located is proportional to the amount of charge on the particle. Thus, it is useful to consider the electrostatic force per charge. We refer to this quantity as the **electric field**, and it is given the symbol $\vec{E}$. From the expression above, we see that the electric field at the point where particle "1" is located is $\vec{F}_1(Net)/q_1$ which is the sum on the right. If we let $\vec{r}_1$ be the position vector to point "1", then

$$\vec{E}(\vec{r}_1) = \sum_{i=2}^{\infty} k \frac{q_i}{r_{1i}^2} \hat{r}_{i1} \tag{8}$$

where $i$ sums over all point particles in the system.

In the above equation for $\vec{E}$, $\vec{r}_1$ is the point in space where particle "1" was located. However, we can define the electric field at this point without particle "1" even being there. The electric field can be defined at any point in space, nothing needs to be there! We usually label the point simply as $\vec{r}$ without the index "1". $\vec{E}(\vec{r})$ is the force per charge **if** a charge were to be placed at the point $\vec{r}$.

The electric field is somewhat abstract. It is similar to the price per weight of a quantity, say bananas. Let the price of bananas be 50 cents per pound. This value, 50 cents/pound, can exist even if no one ever buys bananas. However, **if** someone wants to buy bananas, the bananas are weighed and the price determined. The electric field can be defined at a point in space even if a charged particle is never placed there. However, **if** a particle which has an amount of charge $q$ is placed at the point $r$, **then** the electrostatic force that the particle will feel is

$$\vec{F}_{electrostatic} = q\vec{E}(\vec{r}) \tag{9}$$

In this equation, $q$ is a scalar and can be positive or negative. The force $\vec{F}$ is a vector, and thus $\vec{E}(\vec{r})$ is a vector. There is a difference between the two vectors $\vec{F}$ and $\vec{E}$ in the equation above. The force vector $\vec{F}$ is the force on the particle placed at the position $\vec{r}$. If there is no particle, there is no force vector. The electric field vector, on the other hand, whether or not a particle is located at $\vec{r}$. It can be defined at every point in space. Such a quantity is called a **vector field**. *If there is a vector assigned to every point in space, this collection of vectors is called a vector field.* Before we calculate the electric field vectors for different charge distributions, lets comment on some ideas of the electric field.

*Comments on the electric field*

1. As noted, the electric field is a vector field, it can be defined (or can exist) at all points in space. It depends only on the sources of charge, i.e. the charge distribution.

2. The direction of the electric field at $\vec{r}$ is the same as the direction of the force on a positive charge if it is placed at $\vec{r}$.

3. The force on a negatively charged object placed at $\vec{r}$ is in the opposite direction as the direction of $\vec{E}(\vec{r})$.

4. Is the electric field "real", or is it just a mathematical quantity defined for convenience? Which vector is more "real" the force $\vec{F}$ or $\vec{E}$? The way we have defined $\vec{E}$ above, it is just a mathematical quantity. If you continue in physics, you will see that light can be described (classically) by electric (and magnetic) fields. Light seems to be real, so maybe $\vec{E}$ is also. However, experiments in quantum mechanics are consistent with the electric field being related to the probability of finding a photon at a particular location. This interpretation would make one believe that $\vec{E}$ is not

"real". I'll let you decide how you want to think about the electric field. It really doesn't matter what you think, as long as you do.

Let's finish the section on electric fields by calculating the electrostatic field for some interesting charge distributions. After the calculation, we will comment on the result. We start with the simplest possibility: the electric field produced by a small "point" particle that has a net charge of magnitude $Q$. We refer to this object as a **point charge**.

*$\vec{E}$ in the vicinity of a point charge*
Suppose there is a "point" particle located at the origin that has a net charge of $Q$. What is the electric field at all points in space due to this small charge? Consider a location at the position $\vec{r}$ from the origin. If a particle of charge $q$ is placed there, the electrostatic force $\vec{F}$ it feels is given by Coulomb's law:

$$\vec{F} = k\frac{Qq}{r^2}\hat{r} \tag{10}$$

where $\hat{r}$ is a unit vector that points away from the origin. Since the electric field is the force/charge, $\vec{E} = \vec{F}/q$ or

$$\vec{E}(\vec{r}) = k\frac{Q}{r^2}\hat{r} \tag{11}$$

*Comments*
1. We can now use this simple result **and the principle of superposition** to find the electric field produced by any charge distribution. It was a real theoretical breakthrough to come up with a method of finding the electric produced by "complicated" extended objects. The superposition principle is the key to the simplification: the electric field from many sources add like vectors. One needs only to understand the electric field produced by a "point" particle, then integrate over the solid object to find the complete electric field. It turns out that the electric field produced by a point particle is simple: $\sim Q/r^2$. In this way, one can break down a complicated problem into simpler ones.

In lecture we will first consider a collection of "point charges". Below I summarize the results for some continuous (or solid) charge distributions that will be covered in lecture.

*The Electric field on the axis a long thin rod*

Suppose we have a long thin rod of total charge $Q > 0$ and length $l$. Consider a point on the axis of the rod a distance $d$ from one end. See the figure. What is the electric field at the point located a distance $d$ from the end of the rod?

First we divide up the rod into small pieces. Lets divide it up into $N$ equal pieces. Each piece will have a charge $\Delta Q$ and a length $\Delta x$. Consider the electric field at the point due to a small piece of rod a distance $x$ from the end. The magnitude of the electric field, $\Delta E_x$, from Coulomb's law is:

$$\Delta E_x = k\frac{\Delta Q}{(d+x)^2} \tag{12}$$

Since we will integrate over $x$, we need to express the charge of the small piece $\Delta Q$ in terms of $\Delta x$. Since the whole rod has a net charge $Q$ and length $l$, the charge of the small piece is $\Delta Q = Q(\Delta x/l)$. Thus, we have

$$\Delta E_x = k\frac{Q\Delta x}{l(d+x)^2} \tag{13}$$

The final step is to add up the contribution from all the little pieces of the rod as $\Delta x \to 0$. This leads to the integral expression:

$$E_x = \int_0^l k\frac{Q\ dx}{l(d+x)^2} \tag{14}$$

with the result being:

$$E_x = k\frac{Q}{d(d+l)} \tag{15}$$

with the direction of the electric field pointing away from the rod (since $q > 0$).

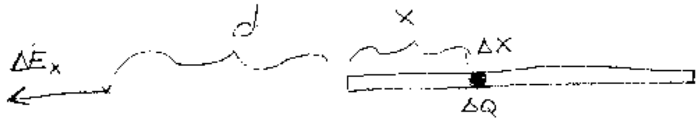
*Comments*

1. One can check the result for the limiting case of $d >> l$. As $d$ becomes much greater than $l$, the term $(d+l)$ is very close to $d$. Thus the limit of $E_x$ as $d \to \infty$ is $kQ/d^2$ which is the electric field produced by a point charge as expected.

2. The ratio $Q/l$ is the charge per length of the rod. This is a kind of density. You normally think of density as mass/volume. $Q/l$ is the charge/length, and is called the **linear charge density**. It is often given the symbol $\lambda \equiv q/l$.

3. Note that the electric field is not equal to $kQ/(d+l/2)^2$:

## ROD




$$\Delta E_x = \frac{k\,\Delta Q}{(x+d)^2}$$

$$\frac{\Delta Q}{Q} = \frac{\Delta x}{\ell}$$

$$\Delta Q = \frac{Q}{\ell}\,\Delta x$$

$$\Delta E_x = \frac{kQ}{\ell}\,\frac{\Delta x}{(x+d)^2}$$

$$E_x = \int_0^{\ell} \frac{kQ}{\ell}\,\frac{dx}{(x+d)^2} = -\frac{kQ}{\ell}(x+d)^{-1}\Big|_0^{\ell}$$

$$E_x = \frac{kQ}{\ell}\left(\frac{1}{d} - \frac{1}{d+\ell}\right) = \boxed{\frac{kQ}{d(d+\ell)}}$$

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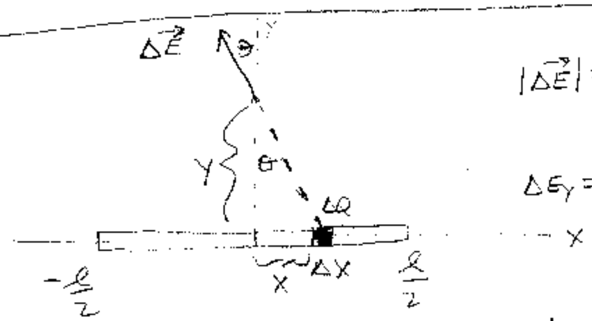



$$|\Delta\vec{E}| = \frac{k\,\Delta Q}{x^2+y^2}$$

$$\Delta E_y = \frac{k\,\Delta Q}{x^2+y^2}\cos\theta$$

but $\cos\theta = \frac{y}{\sqrt{x^2+y^2}}$ and $\frac{\Delta Q}{Q} = \frac{\Delta x}{\ell}$

$$\Delta E_y = \frac{kQ}{\ell}\,\frac{y\,\Delta x}{(x^2+y^2)^{3/2}}$$

$$\boxed{E_y = \frac{kQy}{\ell}\int_{-\frac{\ell}{2}}^{\ell/2} \frac{dx}{(x^2+y^2)^{3/2}}}$$

$$E_x \neq k\frac{Q}{(d+l/2)^2} \tag{16}$$

That is, **in general the electric field is not the same as if all of the objects charge were located at its center of mass**. There is one exception to this: the electric field produced by an object with spherical symmetry. We will consider this case using a clever method discovered by Gauss. But first, some other extended charge distributions.

*The Electric Field at a point on the perpendicular bisector of a long rod*

If time permits we will calculate the electric field at a point that is equal distant from the ends of the rod. Consider a rod of length $l$ with a total charge $Q$ distributed uniformly on it. Lets choose a point "P" that is equal distant from the ends and a distance $y$ from the center of the rod. Let the rod lie on the x-axis with its center at $x = 0$. As before, we divide the rod up into small segments, find the electric field at "P", $\Delta\vec{E}$, due to one of the segments, and add up (integrate) all the segments of the rod (superposition principle).

Consider a segment located at a distance $x$ from the origin having a charge of $\Delta Q$. The magnitude of $\Delta\vec{E}$ at the point "P" due to the segment is

$$|\Delta\vec{E}| = k\frac{\Delta Q}{x^2+y^2} \tag{17}$$

with the direction away from the segment. After integrating over the rod, only the component away from the rod will survive. That is, only $E_y$ will be non-zero. The component of $\Delta\vec{E}$ in the "y" direction, $\Delta E_y$ is given by

$$\begin{aligned}\Delta E_y &= |\Delta\vec{E}|cos\theta \\ &= \frac{ky\Delta Q}{(x^2+y^2)^{3/2}}\end{aligned}$$

As in the previous example, the charge $\Delta Q$ can be expressed in terms of $\Delta x$: $\Delta Q = (Q/l)\Delta x$. $Q/l$ is the charge per length, or the linear charge density. It is often given the symbol $\lambda \equiv Q/l$. Substituting $\Delta Q = \lambda\Delta x$, the integral becomes:

$$E_y = k\lambda y\int_{-l/2}^{+l/2}\frac{dx}{(x^2+y^2)^{3/2}} \tag{18}$$

This integral can be solved analytically, with the result being:

$$E_y = \frac{2k\lambda}{y\sqrt{1 + 4y^2/l^2}} \tag{19}$$

*Comments*
1. An interesting case of this result is to let the length of the rod, $l$, approach infinity. As $l \to \infty$, $E_y$ becomes:

$$E_y = \frac{2k\lambda}{y} \tag{20}$$

Thus, for the infinite charged rod, the electric field decreases inversely with the distance from the rod.

2. We will derive the case of the infinite rod using Gauss' Law later, and obtain the same result.

*The electric field on the axis of a ring*

Suppose we have a ring of total charge $Q > 0$ and radius $a$. Consider a point on the axis of the ring a distance $x$ from the center (See the figure). What is the electric field at the point at $x$ due to the charged ring?

First we divide up the ring into small "point" pieces of charge $\Delta Q$. Next, we find the magnitude of the electric field, $|\Delta\vec{E}|$, at the point due to the small piece of the ring:

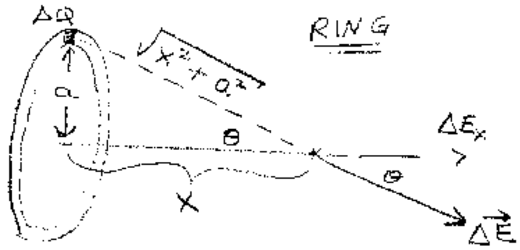
$$|\Delta\vec{E}| = k\frac{\Delta Q}{x^2 + a^2} \tag{21}$$

The direction of the electric field $\Delta\vec{E}$ points away from the edge of the ring where $\Delta Q$ is. Now we need to sum up the forces due to all the little pieces of the ring. This involves an integration: $\int \Delta\vec{E}$. However, for the ring this is simple. Upon integrating around the ring, the only component that survives is the one along the axis: $|\Delta\vec{E}|cos\theta$ or

$$\Delta E_x = k\frac{\Delta Q}{x^2 + a^2}cos(\theta) \tag{22}$$

towards the ring. Since $x$, $a$ and $\theta$ are the same for all the pieces of the ring, the integral is simple:

## RING

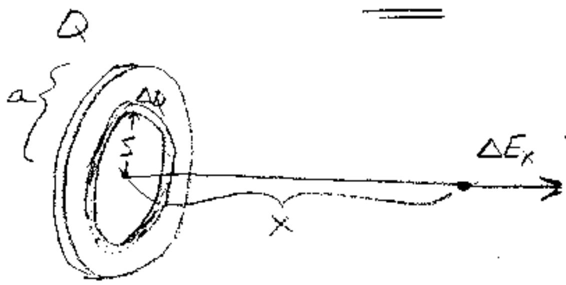



$$|\Delta\vec{E}| = \frac{k\,\Delta Q}{x^2+a^2}$$

$$\Delta E_x = |\Delta\vec{E}|\cos\theta = \frac{k\,\Delta Q}{x^2+a^2}\;\frac{x}{\sqrt{x^2+a^2}} = \frac{k\,x\,\Delta Q}{(x^2+a^2)^{3/2}}$$

$$E_x = \frac{k\,x}{(x^2+a^2)^{3/2}}\sum \Delta Q = \boxed{\frac{k\,Q\,x}{(x^2+a^2)^{3/2}}}$$

## DISK




$$\Delta E_x = \frac{k\,\Delta Q\,x}{(x^2+s^2)^{3/2}}$$

$$\frac{Q}{\pi a^2} = \frac{\Delta Q}{2\pi s\,\Delta s} \quad\Rightarrow\quad \Delta Q = \frac{2Q\,s\,\Delta s}{a^2}$$

$$\Delta E_x = \frac{2k\,x\,Q\,s\,\Delta s}{a^2(x^2+s^2)^{3/2}}$$

$$E_x = \int_0^a \frac{2k\,x\,Q\,s\,ds}{a^2(x^2+s^2)^{3/2}} = \boxed{\frac{2kQ}{a^2}\left(1 - \frac{x}{\sqrt{x^2+a^2}}\right)}$$

$$E_x = k\frac{1}{x^2+a^2}cos(\theta)\int \Delta Q \tag{23}$$

The integral over $\Delta Q$ is just $Q$, so

$$E_x = k\frac{Q}{x^2+a^2}cos(\theta) \tag{24}$$

away from the ring. Since $cos\theta = x/\sqrt{x^2+a^2}$, we have

$$E_x = k\frac{Qx}{(x^2+a^2)^{3/2}} \tag{25}$$

with the direction away from the ring.

*Comments*
1. We will derive this result later by a different method: solving for the electric potential and differentiating it. The result will be the same.

*Electric field on the axis of a circular disk*

As a final example of integrating over an extended charge distribution, we will consider calculating the electric field on the axis of a thin circular disk that is uniformly charged. Suppose we have a very thin disk of radius $a$ that has a total charge of $Q$ uniformly distributed on its surface. We are interested in finding the electric field at a point on the axis a distance $x$ from the center.

We can divide the disk into a series of concentric rings. We just derived an expression for the electric field on the axis of a ring, so we can just add up the contributions of the concentric rings that make up the circular disk. Let $s$ be the radius of a concentric ring with thickness $\Delta s$, and let the ring have a charge of $\Delta Q$. The magnitude of the electric field on the axis due to this ring is

$$E_x = k\frac{\Delta Q x}{(x^2+s^2)^{3/2}} \tag{26}$$

from our example of the ring above. To find the charge of the ring, we use: $(\Delta Q/Q) = (2\pi s\Delta s)/(\pi a^2)$. Substituting above, we have:

$$E_x = \frac{2\pi k Q x s \Delta s}{\pi a^2(x^2+s^2)^{3/2}} \tag{27}$$

Integrating the rings over the whole disk we have:

$$E_x = \frac{2\pi k Q x}{\pi a^2} \int_0^a \frac{s\ ds}{(x^2+s^2)^{3/2}} \tag{28}$$

Solving the integral yields:

$$E_x = \frac{Q/(\pi a^2)}{2\epsilon_0}(1 - \frac{x}{\sqrt{x^2+a^2}}) \tag{29}$$

The ratio $Q/(\pi a^2)$ is the charge per area of the disk, and is also a type of charge density. In this case it is an area density. The symbol $\sigma$ is often used to denote charge/area, **the surface charge density** $\sigma \equiv \Delta Q/\Delta A$. With this definition, we have

$$E_x = \frac{\sigma}{2\epsilon_0}(1 - \frac{x}{\sqrt{x^2+a^2}}) \tag{30}$$

We looked at this example for two reasons. One, to introduce the idea of an area charge density ($\sigma$). The second is to show what happens in the case of an infinitly large disk.

*Comments*

1. In this case of the infinitly large disk, $a \to \infty$. We then have

$$E_x = \frac{\sigma}{2\epsilon_0} \tag{31}$$

for the magnitude of the electric field a distance $x$ from an infinitly large thin disk. This is a very interesting result, it means that the electric field is constant near an infinitely large plane of charge.

2. The above result tells us how to make a constant electric field: uniformly distribute charge on a very large plate.

3. We will derive the case of the infinite plane of charge again using Gauss' Law, and obtain the same result.

## Flux in General

An important mathematical quantity for the formulation of the laws of electrodynamics is flux. We will first define what flux is, then see how we can recast Coulomb's

Law and the superposition principle for electrostatics in terms of the flux of the electric field.

In order to define flux, one needs two things:

1. *A vector field*: A vector field is a "field of vectors". That is, at every point in space there is a vector defined. We have seen one so far, the electric field $\vec{E}$. We will encounter another one this quarter, the magnetic field $\vec{B}$. Other vector fields in physics include the velocity vector in a fluid and the graviational field to name a few.

2. *A surface*. The surface can be a flat surface, a curved surface, or a closed surface.

We will define flux as a scalar quantity. The vector field is a vector. So to make a scalar out of a vector, we will need to use the scalar (or dot) product with another vector. We will do this by associating a vector with the surface. This will be possible if the surface is flat, or very small such that it can be considered flat. Consider a flat surface. **We define the area vector associated with the flat surface as a vector whose direction is perpendicular to the surface and whose magnitude equals the area of the surface**. Last quarter, we also defined the area vector associated with a surface the same way.

We will first define flux for a constant vector field and a flat surface. Suppose there exists a constant vector field $\vec{V}$. This means that at every point in space there is a vector $\vec{V}$, the same vector at all points in space. Also suppose there is a flat surface that has an area $A$. The area vector $\vec{A}$ associated with this surface is perpendicular to the surface. The flux, usually labeled as $\Phi$, is defined as:

$$\Phi \equiv \vec{V} \cdot \vec{A} \tag{32}$$

Remember this definition holds for a constant vector field and a flat surface. Flux is a scalar. The dot product is maximized when the two vector fields are parallel to each other. Thus, the flux will be maximized when the vector field is perpendicular to the surface. The dot product is zero when the two vector fields are perpendicular to each other. Thus, the flux will be zero when the surface is "parallel" to the vector field. Qualitatively, one can think of the flux as how much of the vector field passes through the surface. If we let $\theta$ be the angle between the area vector and the vector field,

$$\Phi \equiv |\vec{V}||\vec{A}|cos(\theta) \tag{33}$$

for a constant vector field and a flat surface. How do we generalize the definition of flux to include a vector field that can vary from point to point in space and to a surface that is not necessarily flat? We can do this by dividing up the surface $S$ into very small surfaces, $\Delta S_i$ where $i$ is the i'th little surface piece. If the $\Delta S_i$ are small enough, they will be very close to being a flat surface. The smaller they are, the closer they will be to being flat. Also, if the surface is small enough, the vector field will be approximately constant on it. Thus, if $\vec{V}_i$ is the vector at the center of the i'th surface, we can define the flux of the vector field $\vec{V}(\vec{r})$ over the surface $S$ as

$$\Phi_V = \lim_{\Delta S_i \to 0} \sum_i \vec{V}_i \cdot \Delta \vec{A}_i \tag{34}$$

where $\Delta \vec{A}_i$ is the area vector associated with the i'th surface. Thus, the limiting process allows us to extend our definition of flux for a flat surface to a curved one. As the small surfaces $\Delta S_i$ get smaller and smaller, the surfaces approach flat ones and the vector field approaches a constant one over the surface. The number of surfaces goes to infinity and the sum becomes an integral. This kind of integral is called a **surface integral** and is written as

$$\begin{aligned}\Phi_V &= \lim_{\Delta S_i \to 0} \sum_i \vec{V}_i \cdot \Delta \vec{A}_i \\ &= \iint_S \vec{V}(\vec{r}) \cdot d\vec{A}\end{aligned}$$

The two integral signs denote that it is a two dimensional integral over the surface $S$. Often $\vec{V}(\vec{r})$ is written simply as $\vec{V}$. However, one must remember that $\vec{V}$ is a vector field and in general can depend on the location of the surface. The subscript $V$ on $\Phi$ reminds us that the flux is for the vector field $\vec{V}$. Remember, when discussing flux, one needs to specify the **vector field** and the **surface**. The above integral can be complicated. In this course, we will only consider surface integrals that are easy to compute. We will usually choose our surfaces such that the vector field is perpendicular to the surface, and has a simple dependence over the surface.

You might be wondering why we are introducing the concept of flux. So far we have considered electrostatics, and haven't needed to use flux in our description. However, when we investigate changing electric (and magnetic) fields, flux will be useful for describing the physics. In the next section we will restate Coulomb's law using flux, via Gauss' Law. Although Gauss' Law does not add any new "physics", it helps in solving some problems involving conductors and symmetric charge distributions. It also is a good introduction to the concept of flux.

## Electric Flux

Electric flux, $\Phi_E$, is the flux for which the vector field is the electric field. The surface must be specified. For electrostatics, the most interesting surface to calculate electric flux is a closed surface. A **closed surface** is a surface that is closed, encloses a volume. For a closed surface there will be an inside and an outside. We choose the direction of the area vector to be **positive** if it points **outward from the surface**.

Suppose there is a "point" particle located at the origin that has a net charge of $Q > 0$. Consider the electric field produced by this charge as the "vector field". For a surface for calculating $\Phi_E$ let's choose a closed spherical surface that has a radius $r$ whose center is at the origin. To denote a closed surface, we write a circle in the integral signs:

$$\Phi_E = \oint\!\!\oint \vec{E} \cdot d\vec{A} \tag{35}$$

Now let's calculate the right side for a spherical surface for the electric field produced by a point charge at the center. Since the electric field is radial from $Q$, $\vec{E}$ is perpendicular to the surface at all points on the surface. Also, since all points on the surface are the same distance from the charge, the magnitude of the electric field on the surface is constant. So the integral is very simple. It is just the magnitude of the electric field times the surface area:

$$\Phi_E = |\vec{E}| 4\pi r^2 \tag{36}$$

Since $\vec{E}$ is produced from a point charge, we have from Coulomb's Law, $|\vec{E}| = kQ/r^2$:

$$\Phi_E = \frac{kQ}{r^2} 4\pi r^2 \tag{37}$$

The $r^2$ cancel, and we are left with

$$\Phi_E = 4\pi k Q \tag{38}$$

Usually one substitutes for $k$: $k = 1/(4\pi\epsilon_0)$,

$$\Phi_E = \frac{Q}{\epsilon_0} \tag{39}$$

Note that in this expession, the radius of the spherical surface does not enter. That is, no matter how big or small $r$ is the electric flux through the spherical surface is $Q/\epsilon_0$.

You might think that this simple result would only apply to the flux through a closed spherical surface with a charge $Q$ at the center. However, we will show what Gauss did, that **even if the surface is not a spherical one the result is the same as long as the charge is within the surface**. This is because of the inverse square fall-off of the electric field with distance from the point source. In lecture we will show that if one considers wedges eminating from the charge that the flux through any surface intersecting the wedge will be the same. Thus, the above equation is valid for any surface that encloses the point charge:

$$\Phi_E = \frac{Q}{\epsilon_0} \tag{40}$$

**for any closed surface that encloses the point charge**. If there is more than one charge within the surface then we can use the principle of superposition. Since the electric fields add like vectors, the electric flux through a closed surface will just be the sum of all the charge within the surface divided by $\epsilon_0$:

$$\Phi_E = \frac{Q_{Net\ inside\ closed\ surface}}{\epsilon_0} \tag{41}$$

This is known as **Gauss' Law**. One often writes explicitly the integral expression for the flux:

$$\oint\!\!\oint \vec{E} \cdot d\vec{A} = \frac{Q_{Net\ inside\ closed\ surface}}{\epsilon_0} \tag{42}$$

Gauss' Law can also be stated in words: The electrical flux (flux for the electric vector field) for any closed surface is equal to the net charge enclosed by the surface. Before we do examples of Gauss' Law for conductors and symmetric charge distributions we mention a few comments.

*Comments on Gauss' Law*:
1. As previously stated, Gauss's Law is just another way of expressing Coulomb's Law and the superposition principle. This is because Gauss's law is only true if the vector field decreases as $1/r^2$ from a point source.

2. Gauss's Law is true for any closed surface. The right side of the equation is easy to evaluate, since one just adds up all the charge within the surface. The left side of the equation can be very difficult to evaluate, since one must do a surface integral. However, if one can choose a surface for which the electric field is simple then the left side can be determined. In the applications for Gauss' Law, we will seek out simple

situations for which the left side can be evaluated.

3. The surface one chooses to apply the equation above is just a mathematical surface. There doesn't need to be anything on it. The closed surface that one chooses for evaluating the electric flux is often called a "Gaussian Surface".

4. Gauss' Law relates the electric field on a closed surface with the charge inside the surface. It is a geometrical approach for describing the electrostatic force.

**Examples using Gauss' Law**

Gauss's Law relates the surface integral of the electric field for a closed surface (electric flux) to the charge within the surface:

$$\oint\oint \vec{E} \cdot d\vec{A} = \frac{Q_{Net\ inside\ closed\ surface}}{\epsilon_0} \tag{43}$$

Gauss's law is true for **any closed surface**, however the flux integral on the left is often very difficult to evaluate. In this class we will apply the above equation to situations for which the left side is "not too difficult" to determine. The trick is to pick a situation for which there exists a surface for which the electric field is simple. This will be true for certain symmetric charge distributions as well as fields within a conductor.

*Infinitely Long (and infinitely thin) rod of charge*

Consider the electric field produced by an infinitely long (thin) rod. Assume that the rod is non-conducting and is uniformly charged. We can describe the amount of charge in terms of the **linear charge density** $\lambda$. Since the rod is infinitely long it has an infinite amount of charge. However the charge/length is a finite number.

What properties does the electric field have?

a) Since the rod is infinitely long, the electric field must point radially away from the rod. It cannot have any component along the direction of the rod, since each direction is equivalent. It cannot have any component circulating the rod since there is no preference for clockwise versus counter-clockwise.

b) The magnitude of the electric field can only depend on the radial distance from the rod.

Because of these properties, if we choose a surface that is a "can" whose axis coincides with the rod, then the electric flux integral is easy to evaluate. Let the "can" have a radius $r$ and a height $L$. We need to calculate the electric flux over the whole closed surface. There will be three parts: one over the sides of the "can", one over the right end and one over the left end:

$$\oint\!\!\oint \vec{E} \cdot d\vec{A} = \int (sides) + \int (right) + \int (left) \tag{44}$$

The flux integral over the ends of the "can" equal zero. This is because the electric field is parallel to the ends of the "can". The flux integral over the side of the "can" just equals the area of the side times $E(r)$ since $\vec{E}$ is perpendicular to the surface:

$$\oint\!\!\oint \vec{E} \cdot d\vec{A} = 2\pi r L E(r) + 0 + 0 \tag{45}$$

The amount of charge within the surface is just $\lambda L$, so Gauss' Law yields:

$$2\pi r L E(r) = \frac{\lambda L}{\epsilon_0} \tag{46}$$

Solving for $E(r)$ we have:

$$E(r) = \frac{\lambda}{2\pi\epsilon_0 r} = \frac{2k\lambda}{r} \tag{47}$$

*Thin shell which is uniformly charged*

Consider the electric field produced by a thin shell of radius $R$ that has a total charge of magnitude $Q$ distributed uniformly on its surface.

What properties does the electric field have?

a) Since the charge distribution is spherically symmetric, the direction of the electric field will be in the radial direction. That is, it must point away from or towards the center of the shell.

b) Since all directions are equivalent, the magnitude of the electric field can only depend on $r$, where $r$ is the distance to the center of the shell. That is, at the position $\vec{r}$: $\vec{E}(\vec{r}) = E(r)\hat{r}$. Here $E(r)$ is the magnitude of the vector field a distance $r$ away from the center.

Thus, if we choose a spherical surface of radius $r$ centered at the shell, then the flux integral is easy to evaluate. Since $\vec{E}$ is perpendicular to the surface with a constant magnitude:

$$\oint\!\!\oint \vec{E} \cdot d\vec{A} = E(r) 4\pi r^2 \tag{48}$$

If $r > R$, Gauss' Law gives

$$E(r) 4\pi r^2 = \frac{Q}{\epsilon_0} \tag{49}$$

Since all the charge is within the surface. Solving for $E(r)$ we have

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2} = k\frac{Q}{r^2} \tag{50}$$

If $r < R$ then there is no charge within the surface and we have

$$E(r) 4\pi r^2 = 0 \tag{51}$$

Or,

$$E(r) = 0 \tag{52}$$

for $r < R$. That is, the electric field inside of the uniformly charged shell is zero.

*Uniformly Charged non-conducting Sphere*

Consider the electric field produced by a uniformly charge non-conducting sphere of radius $R$. By non-conducting we mean that the charge doesn't move. The charge is uniformly distributed throughout the sphere. Let the total charge on the sphere be $Q$. The charge density is $\rho = q/V$. Since the volume of a sphere is $(4/3)\pi r^3$, the charge density in the sphere is $\rho = (3Q)/(4\pi r^3)$.

What properties does the electric field have?
The same as those for the spherical shell just discussed:

a) Since the charge distribution is spherically symmetric, the direction of the electric field will be in the radial direction. That is, it must point away from or towards the center of the shell.

b) Since all directions are equivalent, the magnitude of the electric field can only depend on $r$, where $r$ is the distance to the center of the shell. That is, at the position $\vec{r}$: $\vec{E}(\vec{r}) = E(r)\hat{r}$. Here $E(r)$ is the magnitude of the vector field a distance $r$ away from the center.

Thus, as before if we choose a spherical surface of radius $r$ centered at the sphere, then the flux integral is easy to evaluate. Since $\vec{E}$ is perpendicular to the surface with a constant magnitude:

$$\oint\!\!\oint \vec{E} \cdot d\vec{A} = E(r)4\pi r^2 \tag{53}$$

If $r > R$, Gauss' Law gives

$$E(r)4\pi r^2 = \frac{Q}{\epsilon_0} \tag{54}$$

Since all the charge is within the surface. Solving for $E(r)$ we have

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2} = k\frac{Q}{r^2} \tag{55}$$

This is the same result as for the spherical shell.

Inside the sphere, $r < R$, the same symmetry applies, and $\vec{E}(\vec{r}) = E(r)\hat{r}$. If we choose a spherical surface centered at the center of the sphere but **inside** the sphere, the flux integral is still

$$\oint\!\!\oint \vec{E} \cdot d\vec{A} = E(r)4\pi r^2 \tag{56}$$

However, for this mathematical surface all the charge $Q$ is not enclosed by the spherical surface. The charge inside the surface is:

$$Q_{inside} = \frac{4}{3}\pi r^3 \rho = Q\frac{r^3}{R^3} \tag{57}$$

Applying Gauss' Law we have

$$E(r)4\pi r^2 = \frac{Qr^3}{R^3\epsilon_0} \tag{58}$$

Solving for the electric field ($E(r)$ inside a uniformly charged non-conducting sphere we have

$$E(r) = \frac{kQr}{R^3} \tag{59}$$

*Uniformly Charged infinite plane*

Consider the electric field produced by an infinite "sheet" of non-conducting material that has charge distributed uniformly on it. For example, imagine an infinitely large sheet of thin paper with extra electrons (or protons) distributed uniformly over it. Since the sheet is infinite, it has an infinite amount of charge. However, the charge per area (charge/area) is a finite number. We call the charge per area the **surface charge density**. It is usually given the symbol $\sigma$ in most textbooks.

What properties does the electric field produced by an infinite sheet of charge produce?

a) Since it is infinite, the electric field at locations on either side must point away from (or towards) the sheet. Suppose the charged sheet lies in the y-z plane. Then the electric field must point in the plus or minus $\hat{i}$ direction.

b) The magnitude of the electric field can only depend on the distance from the sheet. That is, if the charged sheet lies in the y-z plane, the magnitude of $\vec{E}$ can only depend on $x$. Thus, the electric field must be of the form:

$$\vec{E} = E(x)\hat{i} \tag{60}$$

where $E(x)$ is a function that only depends on $x$. Since the left side is the same as the right side of the sheet, symmetry requires that $E(-x) = -E(x)$. That is, if $\vec{E}$ points away from (towards) the sheet on one side it also points away from (towards) the sheet on the other side.

The surface we choose for Gauss' Law is (hopefully) clear. We should choose a "can" whose ends are parallel to the sheet of charge. The "can surface" should be placed such that each end is the same distance from the sheet. Let $x$ be the distance from the end to the sheet, and let the area of the ends of the can be $A$. There are three parts to the surface integral: one over the side and two over the ends (right and left):

$$\oint\!\!\oint \vec{E} \cdot d\vec{A} = \int (side) + \int (right) + \int (left) \tag{61}$$

The flux integral over the side of the can equals zero. This is because the electric field is parallel to the side. The integral over each end is just $E(x)A$, since the electric field is perpendicular to the surface and is constant on the surface.

$$\oint\!\!\oint \vec{E} \cdot d\vec{A} = 0 + E(x)A + E(x)A \tag{62}$$

The amount of charge within the "can surface" is just $\sigma A$. Gauss' Law yields:

$$2E(x)A = \frac{\sigma A}{\epsilon_0} \tag{63}$$

or

$$E = \frac{\sigma}{2\epsilon_0} \tag{64}$$

We can write the magnitude as $E$ and not $E(x)$ because the magnitude of the electric field, $E$, does not depend on $x$. That is, the magnitude of the electric field is the same no matter how close or far the location is from the sheet! An infinite sheet of charge produces a constant electric field. We obtained a similar result using Coulomb's Law for a disk of charge when we let the radius of the disk approach infinity.

*Applications to Conductors*

Gauss' Law can be used to help determine how the charge is distributed on a conductor that is in static equilibrium. Conductors are materials that allow (some) valance electrons to flow freely. Any extra electrons that are added to conductors can also flow freely. Good examples of conductors are metals. If the conductor can come to static equilibrium, the electrons will eventually stop moving. The same will be true if there is a deficiency of electrons. This means that **if a conductor is in static equilibrium, the electric field inside of a conductor is zero**. This has interesting consequences when applying Gauss' Law.

If a surface is contained entirely inside of a conducting material (i.e. metal) then the electric flux through the surface is zero, since $\vec{E}$ is zero on the surface. From Gauss' Law:

$$\oint\!\!\oint \vec{E} \cdot d\vec{A} = \frac{Q_{Net\ inside\ closed\ surface}}{\epsilon_0} \tag{65}$$

Since $\vec{E}$ equals zero on the surface (it is inside the conductor), we have

$$0 = \frac{Q_{Net\ inside\ closed\ surface}}{\epsilon_0} \tag{66}$$

Thus: $Q_{Net\ inside} = 0$. This result can be used to deduce three properties of conductors in static equilibrium:

1. *Any excess charge on a conductor resides on the surface of the conductor.* If we use a Gaussian surface that goes up to the surface of the conductor, there can be no charge inside the conductor. All excess charge must be on the conductors surface. Note: we are talking about two different surfaces: one is the mathematical surface for the flux integral, and the other is the surface of the metal conductor.

2. If there is a cavity inside a conductor and a charge $q$ inside the cavity, then there a net charge of $-q$ is induced on the inside walls of the cavity. This can be shown by choosing the mathematical Gaussian surface around the cavity (and inside the conductor). The net charge within the Gaussian surface must be zero, so there is a net charge of $-q$ on the inside walls of the cavity.

3. The electric field just outside of the surface of a conductor is perpendicular to the conductors surface with a magnitude of $\sigma/\epsilon_0$. This can be shown to be true if one uses as a mathematical Gaussian surface a flat cylindrical surface with area $A$ and small height $h$. One of the surfaces should be located inside the conductor, and the other just outside of the conductor's surface. The only contribution to the electric flux is on the surface just outside of the conductor, since $\vec{E} = 0$ inside the surface. Gauss's Law gives:

$$E_{outside} A = \frac{Q}{\epsilon_0} \tag{67}$$

Solving for $E_{outside}$ gives $E_{outside} = Q/(A\epsilon_0) = \sigma/\epsilon$.

### Energy Considerations in Electrostatics

The work energy theorum was a central idea in determining a potential energy function for a conservative force. We will derive an expression for the electrostatic potential energy of a collection of point particles, then generalize the result. We will see that it is useful to define the potential energy per charge, which will be called the voltage. First a simple example.

Consider two point particles, one with a charge of $Q > 0$, and the other with a charge of $q > 0$. Suppose they are initially separated by a distance $r_i$. Let the particle with charge $Q$ be held fixed, and let the other particle be "pushed" directly away from $Q$ by the electrostatic force to a larger distance $r_f$. How much work is done by the electrostatic force ($W_{electrostatic}$) in pushing the particle from $r_i$ to $r_f$?

$$W_{electrostatic}(r_i \rightarrow r_f) = \int_{r_i}^{r_f} \vec{F} \cdot d\vec{r} \tag{68}$$

where $\vec{F}$ is the electrostatic force between the two point particles. From Coulomb's Law, we have

$$W_{electrostatic}(r_i \rightarrow r_f) = \int_{r_i}^{r_f} k\frac{Qq}{r^2}dr \tag{69}$$

Since $q$ moves directly away from $Q$, the direction of $d\vec{r}$ is in the same direction as the force. This is essentially a one dimensional problem. Carrying out the integration, we have

$$W_{electrostatic}(r_i \rightarrow r_f) = k\frac{Qq}{r_i} - k\frac{Qq}{r_f} \tag{70}$$

Note: that this is the work done by the electrostatic force if the path that $q$ moves is directly away from $Q$. What if $q$ is moved in a circular direction around $Q$ (i.e. with a constant radius $r$)? In this case the direction of $\vec{F}$ is **perpendicular** to the direction of the path. The work done by the electrostatic force will be zero in this case.

Any path connecting $\vec{r}_i$ and $\vec{r}_f$ can be divided up into small line segments. For each small line segment, the work done by the electrostatic force will be the force times the change in radial distance, since any circular path direction does not contribute to the work. Thus, the result of the above equation will be true for **any path** from $\vec{r}_i$ to $\vec{r}_f$. Since the work done is path independent, it is a conservative force. This was the same result we found with the gravitational force, since it was also a central force and decreases as $1/r^2$. Actually, any central force will be conservative.

If $q$ has a speed $v_i$ at $\vec{r}_i$, and a speed $v_f$ at $\vec{r}_f$, then from work-energy theorum:

$$\begin{aligned} Net\ Work &= \Delta K.E. \\ k\frac{Qq}{r_i} - k\frac{Qq}{r_f} &= \frac{mv_f^2}{2} - \frac{mv_i^2}{2} \end{aligned}$$

Moving the initial parameters to the left and the final parameters to the right, we have:

$$\frac{mv_i^2}{2} + k\frac{Qq}{r_i} = \frac{mv_f^2}{2} + k\frac{Qq}{r_f} \tag{71}$$

As $q$ moves in the presence of $Q$ (which is fixed), the expression $mv^2/2+kQq/r$ remains constant. The above combination of terms is conserved. We have been calling this combination mechanical energy, with the first term equal to the kinetic energy, and the second term is defined as the potential energy. The electrical potential energy $U$ of two point charges is:

$$U = k\frac{Qq}{r} \tag{72}$$

*Comments*:

1. Potential energy is a scalar.
2. Potential energy falls off as $1/r$. The force decreases as $1/r^2$, but since potential energy involves the integral of force, one power of $r$ is reduced in the denominator.
3. The reference for zero potential is at $r = \infty$.
4. The expression is also true if $Q$ and/or $q$ are negative. If they have opposite signs, the system has negative potential energy. It takes work to separate them infinitely far apart.

The above expession is also called the electrostatic potential energy of the two particle system. If there are three point particles in the system, the total potential energy is found by adding the contributions from each pair of particles:

$$U = k\frac{q_1 q_2}{r_{12}} + k\frac{q_1 q_3}{r_{13}} + k\frac{q_2 q_3}{r_{23}} \tag{73}$$

where $r_{ij}$ is the distance between particle "i" and particle "j". One can add up the potential energies from each pair because electric forces add like vectors (superposition principle). For example, start with $q_1$ fixed in space. The first term is the energy needed to bring particle "2" a distance $r_{12}$ from particle "1". Now keep particle "1" and "2" fixed and bring in particle "3". Since the force on "3" is the vector sum of the forces due to "1" plus "2" one has the last two terms in the sum. This idea can be extended to any number of point particles.

### Summary

This ends our introduction to electrostatics. All the "physics" is described by: Coulomb's Law for point particles plus the superposition principle. Since the force on a particle is proportional to the charge on it, it was convenient to define the "Force per charge" which we called the electric field. We discussed a different way to state Coulomb's Law: Gauss' Law. We found the electric field for different charge distributions consisting of "point charges" as well as continuous charge distributions. Finally, we used the concept of work to determine the electrostatic potential energy of a collection of point particles. In the next section we will continue to extend the physics of Coulomb's law to define voltage, determine how charge behaves on conductors (capacitors), as well as simple electrical circuits.