

Notes on Oscillations and Mechanical Waves

The topics for the second part of our physics class this quarter will be oscillations and waves. We will start with periodic motion for the first two lectures, with our specific examples being the motion of a mass attached to the end of a spring, and the pendulum. The last six lectures will be devoted to mechanical waves and their properties.

Periodic Motion

Periodic motion is motion that repeats itself. For example, a small object oscillating at the end of a spring, a swinging pendulum, the earth orbiting the sun, etc. are examples where the objects motion "approximately" keeps repeating itself. If an objects motion is periodic, then there is a characteristic time: the time it takes for the motion to repeat itself. This is called the period (of the periodic motion) and is usually given the symbol T :

Period (T): The time for one complete cycle of the periodic motion.

For example, the period of the rotation of the earth about its axis is one day. During the quarter, when classes are in session, the period of our activities is one week. We can also speak of the number of cycles repeated per unit time. This is called the frequency of the periodic motion:

frequency (f): The number of cycles per unit time.

A common unit for frequency is one cycle per second. This is defined as one Hertz (Hz). $1 \text{ Hz} \equiv 1 \text{ cycle/sec}$. If the periodic motion occurs f times per second, then the time for one cycle is $1/f$, so

$$T = \frac{1}{f} \tag{1}$$

or equivalently

$$f = \frac{1}{T} \tag{2}$$

In this class we will mainly be describing periodic motion that occurs in one dimension. For example an object will execute periodic motion along the x- or y-axis.

Suppose that the motion of an object occurs along the x-axis. Then the position of the object is given by its position function $x(t)$. If the motion is periodic, it means that:

$$x(t + T) = x(t) \quad (3)$$

where T is the period of the motion. The function $x(t)$ can in general have any shape as long as $x(t) = x(t + T)$. In some cases, $x(t)$ has a simple oscillating shape, that of a pure sinusoidal form. If the motion is perfectly sinusoidal, i.e. a sin or cos function, then motion is called "simple harmonic motion".

Simple Harmonic Motion

If the motion of an object is a perfect single sinusoidal function, we term the motion "simple harmonic". Let's review the mathematics of simple trig functions. We need just three parameters to describe any single sinusoidal function: the period (T), the amplitude (A), and a parameter related to the position at time $t = 0$. The amplitude of the motion is the maximum value from equilibrium that the object travels.

Suppose the object starts off initially at $x = 0$, has some initial speed in the +x direction and an amplitude A . In this case, the motion of the object is described by:

$$x(t) = A \sin\left(\frac{2\pi t}{T}\right) \quad (4)$$

where T is the period of the motion. One often defines $2\pi/T$ to be the variable ω :

$$\omega \equiv \frac{2\pi}{T} \quad (5)$$

So one often sees the equation for this case of simple harmonic motion written as $x(t) = A \sin(\omega t)$. At time $t = 0$, the position is at $x = 0$ since $\sin(0) = 0$. The velocity of the particle at any time t is found by differentiating $x(t)$:

$$v(t) = \frac{dx}{dt} = A\omega \cos(\omega t) \quad (6)$$

The initial velocity is just $v(0) = A\omega$, and is in the + direction. Actually $A\omega$ is the largest speed that the object ever acquires.

The above equation only applies to simple harmonic motion for which the object starts at $x = 0$. How do we generalize this to the case where the object starts off

at $x = x_0$, where $x_0 \neq 0$? This can be done by adding a phase shift, ϕ measured in radians, to the argument of the sin function:

$$x(t) = A\sin(\omega t + \phi) \quad (7)$$

If $\phi < 0$ then the sin function is shifted to the right. If $\phi > 0$ the sin function is shifted to the left. By adjusting the phase angle one can shift the sin function to match the initial condition. If the initial position is defined as $x(0) \equiv x_0$, then

$$x_0 = A\sin(\phi) \quad (8)$$

or

$$\phi = \sin^{-1}\left(\frac{x_0}{A}\right) \quad (9)$$

The function $x(t) = A\sin(\omega t + \phi)$ can be expressed in an alternative form by expanding the sin function. Using the angle addition formula for sin:

$$\sin(\omega t + \phi) = \sin(\omega t)\cos\phi + \cos(\omega t)\sin\phi \quad (10)$$

Substituting into the equation for $x(t)$ gives:

$$x(t) = (A\cos\phi)\sin(\omega t) + (A\sin\phi)\cos(\omega t) \quad (11)$$

This form for $x(t)$ can be written in a more convenient way. At $t = 0$, $x_0 \equiv x(0) = A\sin\phi$:

$$x(t) = (A\cos\phi)\sin(\omega t) + x_0\cos(\omega t) \quad (12)$$

Differentiating the above equation gives:

$$v(t) = A\omega\cos\phi\cos(\omega t) - x_0\omega\sin(\omega t) \quad (13)$$

The initial velocity $v_0 \equiv v(0) = A\omega\cos\phi$. This gives $A\cos\phi = v_0/\omega$. Substituting into the equation for $x(t)$ and rearranging the terms gives a nice form for $x(t)$ in terms of the initial position and velocity:

$$x(t) = x_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) \quad (14)$$

This concludes our review of the two different forms of a sinusoidal function. Now for the Physics!

What kind of Force Produces Simple Harmonic Motion?

Suppose an object, of mass m , is undergoing simple harmonic motion along the x -axis. This means that the object's position follows a single sinusoidal function, whose general form is $x(t) = A\sin(\omega t + \phi)$. What kind of force produces this kind of motion? Since we know $x(t)$, we can find $a(t)$. The force is just $F(t) = ma(t)$. The velocity is found by differentiating $x(t)$:

$$v(t) = A\omega\cos(\omega t + \phi) \quad (15)$$

Differentiating again for $a(t)$:

$$a(t) = -A\omega^2\sin(\omega t + \phi) \quad (16)$$

However, notice that $x(t) = A\sin(\omega t)$, which simplifies to:

$$\frac{d^2x}{dt^2} = -\omega^2x(t) \quad (17)$$

or

$$a(t) = -\omega^2x(t) \quad (18)$$

or simply $a = -\omega^2x$ at all times and all positions x . This is a simple result: the acceleration is proportional to x , the object's displacement from the equilibrium position. The negative sign means that if $x > 0$ then $a < 0$, and if $x < 0$ then $a > 0$. This means that the acceleration is always towards the origin. Since $F = ma$, we have

$$F = -m\omega^2x \quad (19)$$

This result is also quite simple. The force is always towards the origin, and is proportional to x the distance from the origin. If the object is moved twice as far away from the equilibrium position, it feels twice the force pushing it back. This type of force is called a "linear restoring force". It is a restoring force because of the minus sign. For positive x the force is in the negative direction, and for negative x the force is in the positive direction. It tries to "restore" the object back to $x = 0$ the equilibrium position. It is a linear force because its magnitude is proportional to x the displacement from equilibrium.

Thus we see that **a linear restoring force produces motion that is a single sinusoidal function (simple harmonic motion)**. Let's consider some forces to see if they produce simple harmonic motion, and if so, how the period depends on

the parameter of the force.

The simple spring

We are all familiar with a spring. Suppose an object of mass m is attached to one end, while the other end of the spring is held fixed. The object will have an equilibrium position, call it $x = 0$. If the object is displaced away from $x = 0$ the spring pulls (or pushes) it back to the equilibrium position ($x = 0$). Thus, the spring produces a restoring force. But is the restoring force linear? That is, if we pull the object twice as far away from $x = 0$ is the force twice as much? This question is answered by doing the experiment. You will do measurements in lab that show that **the restoring force of a common spring is approximately linear**. That is:

$$F_{spring} \approx -kx \quad (20)$$

The proportionality constant k is called the spring constant for the particular spring. If k is large, then the spring is stiff and produces a lot of force for a small displacement. If k is small, then the spring is "loose" and doesn't pull back with much force. The spring constant k can be measured by applying a force (F_0) and measuring how much the spring stretches (d). Then $k = F_0/d$. k has units of force/distance (N/m). If time permits, we will discuss different spring examples in class.

Having established that the simple spring produces a linear restoring force (approximately), we can determine period of the resulting periodic (simple harmonic) motion. From the equation above, $F = -m\omega^2x$. Since $F_{spring} = -kx$ we have:

$$k = m\omega^2 \quad (21)$$

or

$$\omega = \sqrt{\frac{k}{m}} \quad (22)$$

Since the period $T = 2\pi/\omega$, the period for the objects motion for the spring force is:

$$T = 2\pi\sqrt{\frac{m}{k}} \quad (23)$$

Although this formula was derived for the simple spring, a similar formula will result in general for a linear restoring force. If the force on an object (moving in one dimension) is of the form $F_x = -Cx$, then the motion will be simple harmonic, $x(t) = A\sin(\omega t + \phi)$ where $\omega = \sqrt{C/m}$.

Notice that in the equation for the period T for the linear restoring force that T only depends on the mass and the restoring force constant. It does not depend on the amplitude A of the motion. Thus, oscillations with large amplitudes will have the same period as oscillations with small amplitudes. This phenomena is a nice characteristic of the linear restoring force. The final equation of motion for the object at the end of the spring is:

$$x(t) = A \sin\left(\sqrt{\frac{k}{m}}t + \phi\right) \quad (24)$$

The Simple Pendulum

A simple pendulum is just a pendulum made out of a small object attached to a light string. Ideally, it is a "point" particle attached to a massless string which is fixed to a pivot point. If the pendulum is displaced from equilibrium, it swings back and forth, and its motion is periodic. The questions we want to consider are: is the motion simple harmonic, and what is the equation for the period T of the motion?

To answer these questions, one starts with the equation relating forces and motion. I am going to use a different variable than the textbook. I will specify the position of the particle by the distance, along an arc, that the particle is from the equilibrium position. See the figure at the end of this section. If the length of the pendulum is l , the ratio x/l is the angle θ (in radians) that the string makes with the vertical. Most textbooks use the angle θ to specify the location of the particle, so $x = l\theta$ is the connection between the text's θ and our x . I think this will be easier for us to make reference to the spring equation and avoid using torque.

When the pendulum is hanging vertical, $x = 0$, right displacement is positive x and left displacement is negative x . If an object is displaced a distance x along the arc, the component of gravity in the direction of the arc is $-mg\sin(\theta)$ or $-mg\sin(x/l)$. The negative sign means that the force of gravity is a restoring force. If $x > 0$, $\sin(x/l) > 0$ and the force is in the negative direction. If $x < 0$, $\sin(x/l) < 0$ and the force is in the positive direction. Thus,

$$F_x = -mg\sin\left(\frac{x}{l}\right) \quad (25)$$

For a force to produce simple harmonic motion, the force must be proportional to $-x$. The equation for the simple pendulum is NOT of this form. Here the force is proportional to $-\sin(x/l)$. Thus, **the motion of a simple pendulum is NOT not simple harmonic**. Don't get confused with the "sin" function. **A sinusoidal**

restoring force does not produce perfect sinusoidal motion. Only a linear restoring force gives perfect sinusoidal motion. This is easy to demonstrate. For simple harmonic motion, the period does not depend on the amplitude. Pendula with a larger amplitude (larger x_{max}) have a longer period than ones with a smaller amplitude (smaller x_{max}).

Approximate Sinusoidal Motion

If x/l is small, less than 0.1, then $\sin(x/l) \approx x/l$. This can be seen from the series expansion of $\sin(x/l)$. The expansion for $\sin(\theta)$ is $\sin(\theta) = \theta - \theta^3/3! + \theta^5/5! - \dots$ (for θ in radians). Replacing $\theta = x/l$ gives: $\sin(x/l) = x/l - (x/l)^3/3! + (x/l)^5/5! - \dots$. If $(x/l) < 0.1$, then the second term is 0.0016 times the first, so to a pretty good approximation $\sin(x/l) \approx x/l$. In this case, we have

$$F_x \approx -\frac{mgx}{l} \quad (26)$$

Thus, for small angles the restoring force is proportional to displacement, and the resulting motion is simple harmonic. The proportionality constant C is mg/l . Since the proportionality constant equals $m\omega^2$, we have $m\omega^2 = mg/l$, or

$$\omega = \sqrt{\frac{g}{l}} \quad (27)$$

For small oscillations, the position x along the arc of the path is just $x(t) = A\sin(\sqrt{\frac{g}{l}}t + \phi)$.

Since $T = 2\pi/\omega$ we have

$$T \approx 2\pi\sqrt{\frac{l}{g}} \quad (28)$$

for a simple pendulum swinging at small displacements x such that x_{max}/l . In terms of the angle that the string makes with the vertical the maximum angle θ_{max} should be less than 0.1 radians.

The differential equation for $x(t)$ can be obtained by using $F_x = md^2x/dt^2$:

$$m\frac{d^2x}{dt^2} = -mg\sin\left(\frac{x}{l}\right) \quad (29)$$

or

$$\frac{d^2x}{dt^2} = -g\sin\left(\frac{x}{l}\right) \quad (30)$$

Using the small angle approximation, we have

$$\frac{d^2x}{dt^2} \approx -\frac{g}{l}x \quad (31)$$

To compare with the equations in the book, just substitute $x = l\theta$.

Energy considerations for Simple Harmonic Motion

To obtain an expression for the total mechanical energy that a simple harmonic oscillator has, we need an expression for the potential energy for the force acting. Let's consider the case of the simple spring. The work done by the spring force when an object moves from the position x_1 to the position x_2 is:

$$W_{1 \rightarrow 2} = \int_{x_1}^{x_2} (-kx)dx \quad (32)$$

One needs to integrate since the force is not constant, but rather changes with position. The integral is:

$$W_{1 \rightarrow 2} = \frac{k}{2}x_1^2 - \frac{k}{2}x_2^2 \quad (33)$$

From the work-energy theorem: $W_{net} = \Delta(K.E.)$. If the only force acting on the object is the force of the spring, then the net work is the work done by the spring. In this case we have:

$$\frac{k}{2}x_1^2 - \frac{k}{2}x_2^2 = \frac{m}{2}v_2^2 - \frac{m}{2}v_1^2 \quad (34)$$

Rearranging the terms gives:

$$\frac{k}{2}x_1^2 + \frac{m}{2}v_1^2 = \frac{k}{2}x_2^2 + \frac{m}{2}v_2^2 \quad (35)$$

All the terms on the left are from position 1, and all the terms on the right are from position 2. Since these positions are arbitrary, we have that the expression $\frac{k}{2}x^2 + \frac{m}{2}v^2$ is a constant of the motion. In this expression, x is the position of the object and v is the speed of the object (at the position x). As the object moves back and forth, x and v both change but the special combination $\frac{k}{2}x^2 + \frac{m}{2}v^2$ remains constant. We recognize the first term to be the potential energy and the second term to be the kinetic energy of the object. The sum, which is the total mechanical energy, remains constant.

$$E_{tot} = \frac{k}{2}x^2 + \frac{m}{2}v^2 \quad (36)$$

To determine the total mechanical energy for simple harmonic motion in general, we need to substitute for $x(t)$ and $v(t)$. In general $x(t) = A\sin(\omega t + \phi)$. The velocity is given by $v(t) = dx/dt = \omega A\cos(\omega t + \phi)$. Substituting into the above equation, we have:

$$E_{tot} = \frac{k}{2}A^2\sin^2(\omega t + \phi) + \frac{m}{2}\omega^2 A^2\cos^2(\omega t + \phi) \quad (37)$$

However, since $m\omega^2 = k$, we have

$$E_{tot} = \frac{k}{2}A^2(\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)) \quad (38)$$

Since $\sin^2 + \cos^2 = 1$, the equation simplifies to

$$E_{tot} = \frac{k}{2}A^2 \quad (39)$$

The total mechanical energy of a mass-spring system is $k/2$ times the square of the amplitude of the motion. In general, if the proportionality constant is C ($F_x = -Cx$), the total mechanical energy is $\frac{C}{2}A^2$.

”Mechanical Waves”: Waves in a Medium

In our previous discussion, we considered oscillations, or periodic motion, in one dimension. That is, an object that moves back and forth in one dimension (e.g. a mass on a spring). Now we want to consider oscillations that take place in a medium. By a **medium** we mean a material like a gas, liquid or solid. **Oscillations that take place in a medium are called mechanical waves.** The word mechanical just means that there is a medium that is oscillating.

What is meant by the expression that a medium is oscillating? It means that the constituents of the medium (the atoms and molecules) are moving back and forth about an average equilibrium position. For the single particle undergoing periodic motion, we described its motion by its position $x(t)$ on the x-axis. Now we have a much different situation to describe. Each point in the medium will have its own displacement as a function of time. I am going to use the symbol d to stand for the displacement of the medium. (The text uses the symbol y which I find confusing.) Each point of the medium will have a displacement that usually varies with time:

$d(\vec{r}, t)$ is defined as the displacement of the medium at the position \vec{r} and at the time t .

There are two "distances" involved here. One is the position \vec{r} in the medium. The other is d , the displacement of the medium at the position \vec{r} . d depends on position and time. It is actually a "field" quantity, a quantity that depends on position and time. To make the analysis simple, here we will only consider changes in the position \vec{r} in one dimension.

For example, suppose the medium is a taut string, like a guitar string. Let the x-axis lie along the string. $d(x, t)$ refers to how much the guitar string is displaced above its equilibrium position at the position x at time t . If the string is not vibrating, then $d(x, t) = 0$ for all x and t . Suppose the string is vibrating. If $x(5 \text{ cm}, 2 \text{ sec}) = 0.5 \text{ cm}$, it means that at the position 5 cm down the string, the string is displaced 0.5 cm at the time 2 seconds. d is different at other positions on the string and as the time changes so will d . It is important to understand what $d(x, t)$ means. We will use the function $d(x, t)$ to describe waves, and will derive an equation that $d(x, t)$ satisfies.

One cannot make a "simple" graph of $d(x, t)$ in two dimensions, since d itself depends on x and t . However, two graphs can be used to describe the displacement. One would be a "snapshot" of the wave at a certain time t , say $t = 0$. In this graph, one would plot $d(x, 0)$ versus x . It would show a picture of *the disturbance at a moment in time*. The other graph would be a graph of the displacement as a function of time at a certain location, say $x = 0$. Here, one would plot $d(0, t)$ as a function of time, and the graph would display the displacement of the medium as a function of time at the position $x = 0$.

Traveling Waves

Under certain conditions, the disturbance (or wave) can move with a definite velocity v . One part of the medium "pushes" on the part next to it, and the disturbance propagates down the medium. We will discuss later why this can occur, but let's first consider how such a traveling wave can be described mathematically. Suppose the wave has the functional form $d = f(x)$ at time $t = 0$. If the wave **travels to the right** with speed v , what is d for times $t > 0$? One just replaces x with $x - vt$:

$$d(x, t) = f(x - vt) \tag{40}$$

for all x and t . If the disturbance (or wave) is **traveling to the left**, then x is replaced with $x + vt$:

$$d(x, t) = f(x + vt) \tag{41}$$

Let's consider an example to better understand this concept. Suppose at $t = 0$ the displacement is

$$d(x, 0) = \frac{1}{x^2 + 1} \quad (42)$$

A graph of this function is given on the figures page. It looks like a mountain that has its largest value of 1 at $x = 0$. To make this shape move to the right with speed 2, we just replace x with $(x - 2t)$:

$$d(x, t) = \frac{1}{(x - 2t)^2 + 1} \quad (43)$$

To see that this shape moves to the right, we can graph $d(x, 1)$ the displacement at time $t = 1$:

$$d(x, 1) = \frac{1}{(x - 2)^2 + 1} \quad (44)$$

This graph has the same "mountain" shape as the previous one, but is shifted two units to the right. It has a maximum of 1 at $x = 2$, since at $x = 2$ the term $(x - 2)$ is zero.

Note that the function $f(x)$ can (in principle) be anything. $f(x)$ does not need to be periodic. Periodic traveling waves are a special case, which we consider next.

Periodic Traveling Waves

If the function $f(x)$ is periodic, then the traveling wave is periodic in position x . If one takes a "snapshot" of the wave at any time t , the disturbance repeats as x is increased (or decreased). The **shortest distance that repeats is called the wavelength** and is usually given the symbol λ .

An interesting property of traveling waves that are periodic in x (at a fixed time t), is that they are also periodic in time for a fixed position x . $d(x, t)$ for x fixed oscillates periodically in time. As with the spring and pendulum, we define the number of oscillations per unit time as the frequency f . Thus, a periodic traveling wave has a definite wavelength λ and a definite frequency f . These two quantities are related to the wave's velocity. If f cycles occur every second, and for each cycle the wave travels a distance λ , then the velocity of the wave is equal to $f\lambda$:

$$v = f\lambda \quad (45)$$

This equation is universal for periodic traveling waves. It holds for sound waves, water waves, waves on a string, and waves on a spring to name a few. It even is true for non-mechanical waves like electro-magnetic radiation (light). Periodic traveling waves have three quantities associated with them: a velocity, wavelength and frequency. General traveling waves only have a velocity associated with them, since the shape need not be periodic.

A periodic traveling wave has some interesting mathematical properties: $d(x \pm \lambda, t) = d(x, t)$ for all times t , and $d(x, t \pm T) = d(x, t)$ for all positions x along the wave. These properties define the periodic traveling wave, the wavelength λ , and the period T of the wave. Combining the two equations gives:

$$d(x \pm \lambda, t) = d(x, t \pm T) \quad (46)$$

Since a traveling wave can be written in the form $d(x \pm vt)$, the above equation yields:

$$(x \pm \lambda) - vt = x - v(t \pm T) \quad (47)$$

for a wave traveling to the right. The x and vt cancel on both sides, and we are left with

$$\lambda = vT \quad (48)$$

using the $+$ sign for λ and the $-$ sign for T . Since $f = 1/T$ we have

$$v = f\lambda \quad (49)$$

Before we talk about a specific kind of periodic traveling wave, the sinusoidal traveling wave, let's discuss some general properties of traveling waves.

Longitudinal versus Transverse Disturbances

The velocity of a traveling wave will have a certain direction. Let's suppose it is traveling in the $+x$ direction. The displacement of the medium can be in any direction. It can be in the x -, the y -, or the z -direction, or a combination of the three directions. We have special names if the displacement is in the direction of the velocity (x -direction) or perpendicular to the direction of the velocity (y - or z -direction).

Longitudinal Waves are traveling waves whose displacements are in the direction of the velocity.

Transverse Waves are traveling waves whose displacements are perpendicular to the direction of the velocity.

Some waves are a combination of both longitudinal and transverse displacements. Let's list some of these properties for some common waves:

Property	string	sound	surface water	E-M (light)
medium:	string	gas, liquid, solid	water	NONE (space and time)
type:	transverse	longitudinal	combination of both	transverse in free space
speed	$\sqrt{F/\mu}$	$\sqrt{B/\rho}$	depth of water	3×10^8 m/s for all observers
frequency		pitch		color for visible light

Developing a Wave Equation

We would like to understand the propagation of a traveling wave in terms of some level of fundamental physics. Some questions we would like to answer are: a) how can the disturbance of a medium travel in space? b) what does the speed of the traveling wave depend upon? We will answer these questions for a simple case: transverse displacements on a taut string. There are basically only three parameters that describe the string: its mass, its length, and the tension in the string.

One could use unit analysis to get an idea how the mass m , length l , and tension affect the speed of the wave. We'll use the symbol F for the tension in the string (T is used for period). F has units of Newtons = mass(length)/(time)². Since we can only use mass and length, we must divide by mass, multiply by length, and take the square root to obtain a velocity: $\sqrt{Fl/m}$. Thus, the speed can only depend on the tension F and the ratio of mass to length m/l . The mass per length is a **linear mass density** and is given the symbol μ . So $v \sim \sqrt{F/\mu}$. Unit analysis can only take us this far. We do not know if there is a unitless factor in front of $\sqrt{F/\mu}$. We need to use more physics.

Consider a small segment of the string, which starts at $x \equiv x_1$ and has a length of Δx and a mass Δm . See the figure on the last page. The right side of the segment is at $x + \Delta x \equiv x_2$. From the figure, the force on the segment at x_2 equals $F \sin \theta_2$ and is away from equilibrium. The force on the segment at x_1 equals $F \sin \theta_1$ and is towards equilibrium. The net force on the segment is therefore $F_{net} = F \sin \theta_1 - F \sin \theta_2$. If the angle is small, $\sin \theta \approx \tan \theta$. But, $\tan \theta$ is equal to the derivative of d with respect to x . So the net force on the segment is

$$Net\ Force \approx F \frac{d'' d''}{dx} \Big|_{x+\Delta x} - F \frac{d'' d''}{dx} \Big|_x \quad (50)$$

Equating Net Force to mass times acceleration gives:

$$\Delta m \frac{d^{2''} d''}{dt^2} = F \frac{d'' d''}{dx} \Big|_{x+\Delta x} - F \frac{d'' d''}{dx} \Big|_x \quad (51)$$

Dividing by Δx on both sides gives

$$\frac{\Delta m}{\Delta x} \frac{d^{2''} d''}{dt^2} = F \frac{\frac{d'' d''}{dx} \Big|_{x+\Delta x} - \frac{d'' d''}{dx} \Big|_x}{\Delta x} \quad (52)$$

Taking the limit as $\Delta x \rightarrow 0$, the equation becomes

$$\frac{d^{2''} d''}{dt^2} = \frac{F}{\mu} \frac{d^{2''} d''}{dx^2} \quad (53)$$

This equation is called a "wave equation". It contains derivatives in space and time for the function $d(x, t)$. This looks like a difficult differential equation to solve, but surprisingly, the general solution is fairly simple. The solution can be written in the form: $d(x, t) = g(u)$ where $u = x \pm vt$ and $g(u)$ is any function of u that has a non-zero second derivative with respect to u . The "-" produces a disturbance (or wave) traveling to the right, and a "+" is a wave traveling to the left. Let's show why $g(u)$ is a solution. I will let $g' \equiv dg/du$ and $g'' \equiv d^2g/du^2$. Consider the case for which $u = x - vt$. $d'' d'' / dx$ is given by

$$\frac{d'' d''}{dx} = \frac{dg}{du} \frac{du}{dx} = g' \quad (54)$$

using the chain rule. The second derivative is determined in the same way:

$$\frac{d^{2''} d''}{dx^2} = \frac{dg'}{du} \frac{du}{dx} = g'' \quad (55)$$

We can use the chain rule in a similar way with the time derivatives:

$$\frac{d'' d''}{dt} = \frac{dg}{du} \frac{du}{dt} = (-v)g' \quad (56)$$

The $-v$ comes from differentiating $u = x - vt$ with respect to t . Taking the second derivative with respect to time gives:

$$\frac{d^{2''} d''}{dt^2} = \frac{dg'}{du} \frac{du}{dt} = v^2 g'' \quad (57)$$

Comparing the second derivative equations for x and t , we see that

$$\frac{d^{2''} d''}{dt^2} = v^2 \frac{d^{2''} d''}{dx^2} \quad (58)$$

Thus, any function of the form $d = g(x - vt)$ which has non-zero second derivatives will satisfy the wave equation, with the velocity of the wave equal to $v = \sqrt{F/\mu}$.

Let's consider a specific example to better understand this concept. Since $g(u)$ can be anything, let's let $g(u) = u^3$. This means that $d(x, t)$ is $(x - vt)^3$:

$$d(x, t) = (x - vt)^3 \quad (59)$$

If we take the second derivative of d with respect to x we get $6(x - vt)$ using the chain rule. The second derivative of d with respect to t is $6v^2(x - vt)$. Thus,

$$\frac{d^{2''} d''}{dt^2} = 6v^2(x - vt) \quad (60)$$

However, $6(x - vt) = d'' d'' / dx^2$. Substituting this into the above equation, we have

$$\frac{d^{2''} d''}{dt^2} = v^2 \frac{d^{2''} d''}{dx^2} \quad (61)$$

Thus, $d(x, t) = (x - vt)^3$ is a solution to the wave equation with $v = \sqrt{F/\mu}$. We would have had the same result if we had used any function $g(u)$ with $u = x - vt$.

This is a nice result and deserves some comments:

1. We applied Newton's law of motion to the segment of string, with the resulting motion that of a traveling wave: if a pulse is formed on the string, it keeps its shape and travels down the string. It is not obvious that this should occur. What was the key property of the medium that lead to a "fixed shape" traveling down the string? From the wave equation, we see that it was important that the second derivative of d in space is proportional to the second derivative in time. The second derivative in time is the acceleration of the segment which is proportional to the force. So the key property is that the force on a segment of the medium is proportional to the second derivative of d . We shall see this same property when we analyze a sound wave.
2. The velocity only depends on the tension F and the linear mass density μ of the string. **The velocity does not depend on the frequency of the disturbance.**

If a medium has this property it is called a **non-dispersive medium**. In a non-dispersive medium a pulse can travel down the medium without its shape changing. f times λ equals v , which is the same for all frequencies, as long as the tension and density do not change. Thus if the frequency increases, the wavelength decreases with the product being constant. Air is non-dispersive for sound waves. This is fortunate, because if the speed of sound depended on the pitch of the note, you wouldn't want to listen to music.

Sinusoidal Traveling Waves

Sinusoidal traveling waves are traveling waves that have a perfect sinusoidal function. Their shape is given by $d = A\sin(2\pi u/\lambda)$ where λ is the wavelength and A is the amplitude of the wave. If they are traveling to the right, then $u = x - vt$. If they are traveling to the left, then $u = x + vt$. Substituting these expressions for u into the sinusoidal function we have

$$d(x, t) = A\sin\left(\frac{2\pi}{\lambda}(x \pm vt)\right) \quad (62)$$

where the + (-) refers to waves traveling to the left (right). Since $v/\lambda = f$, this can be written as:

$$d(x, t) = A\sin\left(\frac{2\pi}{\lambda}x \pm 2\pi ft\right) \quad (63)$$

The expression can be made simpler if we define $\omega \equiv 2\pi f$ and $k \equiv 2\pi/\lambda$. k is called the **wave number**.

$$d(x, t) = A\sin(kx \pm \omega t) \quad (64)$$

. If $d(0,0)$ is not zero, then we need an additional constant to match the initial conditions. As in the case of simple harmonic motion in one dimension, we use a phase ϕ to "put" the sine function in the correct position at $t = 0$:

$$d(x, t) = A\sin(kx \pm \omega t + \phi) \quad (65)$$

The "-" corresponds to waves traveling to the right (+x direction) and the "+" to waves traveling to the left (-x direction).

Sound Waves

The derivation of the wave equation for sound waves is similar to the string. In the case of sound waves the speed is equal to $v_{sound} = \sqrt{B/\rho}$, where B is the bulk modulus and ρ is the density of the medium. The speed of sound is quite different in gases, liquids, and solids. For example, in air $v_{sound} \approx 340$ m/s, in water $v_{sound} \approx 1500$ m/s, and in solids it can be as large as 5000 m/s. Of the many topics we could cover on sound, we will discuss two: the Doppler effect and Sound Level Intensity.

The Doppler Effect for Sound Waves

Suppose there is a source of sound which emits waves of a definite frequency f_s , with the s standing for source. If the source is moving in the medium, the crests of the sound waves will be closer together in front of the source and further apart behind the source than if the source were at rest in the medium. An observer stationary in the medium will hear a frequency different than f_s . This same effect will occur if the source is at rest in the medium and the observer is moving. A change in the frequency of the observed sound from f_s is called the Doppler effect, named after Doppler who first observed it.

Moving Source

Let f_s be the frequency of the source of the sound and f_o be the frequency observed by an observer at rest in the medium. Let v_s be the velocity of the source with respect to the medium, and v the velocity of sound in the medium. Suppose the source is moving towards an observer who is stationary in the medium. The frequency that the observer will hear will be higher than f_s . We will derive in class that:

$$f_o = \frac{f_s}{1 - v_s/v} \quad (66)$$

The minus sign in the denominator makes $f_o > f_s$. If the source is moving away from the observer the result is

$$f_o = \frac{f_s}{1 + v_s/v} \quad (67)$$

and $f_o < f_s$. Both of these equations can be combined into one:

$$f_o = \frac{f_s}{1 \pm v_s/v} \quad (68)$$

The key idea for deriving this equation is to note that the wavelength of the sound in the medium equals the "normal" wavelength minus $v_s T$ where T is the period of the wave.

Moving Observer

Suppose the source is at rest in the medium and an observer is moving towards the source. The observer will observe a higher frequency than f_s . Let v_o be the velocity of the observer. It will be shown in lecture that

$$f_o = f_s(1 + v_o/v) \quad (69)$$

for an observer moving towards a source at rest in the medium. If the observer is moving away from the source

$$f_o = f_s(1 - v_o/v) \quad (70)$$

In this case $f_o < f_s$. These two equations can be combined into one:

$$f_o = f_s(1 \pm v_o/v) \quad (71)$$

General Doppler Shift Equation

Both situations of moving source and observe can be expressed in one equation:

$$f_o = f_s \frac{1 \pm v_o/v}{1 \pm v_s/v} \quad (72)$$

The key to using this equation properly is to use all variables **in the reference frame of the medium**. That is, in the frame with the medium at rest. When determining f_o , the frequency heard by the observer, I substitute into the equation for f_s , the frequency of the source, v_o/v , and v_s/v without assigning a "+" or "-" in the numerator or demonimator. Then I think:

a) If the *observer is moving* towards the source, f_o should increase \rightarrow a "+" in the numerator. If the *observer is moving* away from the source, f_o should decrease \rightarrow a "-" in the numerator.

b) If the *source is moving* towards the observer, f_o should increase \rightarrow a "-" in the denominator. If the *source is moving* away from the observer, f_o should decrease \rightarrow

a "+" in the denominator.

If both are moving with respect to the medium, then apply both a) and b). above.

Sound Intensity

Sound intensity, I , is a measure of the Energy per Area per sec of the sound wave. The metric units of sound intensity is Joules/m²/sec. The human ear is rather amazing. It can detect sound intensity as small as 10⁻¹² W/m² and as large as 1 W/m². That is 12 orders of magnitude!! For this reason it is useful to have a different way to assign a number to sound intensity. It is called the sound level and the unit given is the decibel. The sound level in dB is defined in terms of the sound intensity I as:

$$\text{Sound Level (dB)} \equiv 10\log\left(\frac{I}{10^{-12}}\right) \quad (73)$$

where the sound intensity I is in units of W/m², and the log is in base 10. The table below gives some examples of sound levels for various everyday activities:

Activity	I W/m ²	$I/10^{-12}$	$\log(I/10^{-12})$	dB
Threshold of hearing	10 ⁻¹²	1	0	0
Whisper at 1 Meter	10 ⁻¹⁰	10 ²	2	20
Classroom	10 ⁻⁷	10 ⁵	5	50
Jackhammer	10 ⁻³	10 ⁹	9	90
Rock Band	10 ⁻¹	10 ¹¹	11	110
Pain	1	10 ¹²	12	120

Inverse Square Law

Usually sound sources produce a certain amount of sound energy per unit time. That is their output is usually measured in terms of their power. Power is energy per unit time, with the metric unit being Watts. One Watt equals one Joule/sec. In general, the further away one is from a source of sound, the smaller is the sound intensity. If the source of sound is a "small" source and the waves travel away equally in three dimensions the sound intensity is easy to determine. If the observer is a distance r away from the source, the energy distributes itself equally on a spherical surface of radius r . Thus, the sound intensity from a source of power P is just:

$$I = \frac{P}{4\pi r^2} \quad (74)$$

If one moves twice as far away from a sound source, the intensity decreases by a factor of 2^2 or 4.

Superposition Principle

Suppose there are two or more sources of mechanical waves. What is the net displacement of the medium at a particular position? As verified by experiment, the net displacement is the sum of the displacements from the individual sources. This property of nature is called the superposition principle. Simply stated it is:

When two or more waves are present simultaneously at the same place, the resulting displacement is the sum of the displacements from the individual sources.

In this course, we will consider the case when there are only two sources, and each source emits pure sinusoidal waves of equal amplitude.

Two Sources of Sinusoidal Waves

For two sources of waves, the net displacement will be the sum of the displacements from the two sources: $d_{net} = d_1 + d_2$. If the sources emit sinusoidal waves, d_{net} will be the sum of two sinusoidal waves. We will need to use the formula for the sum of two sin functions with equal amplitude:

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \quad (75)$$

In this course we will consider three different cases with two sources: a) two sources separated in space, b) two sources of slightly different frequency, and c) standing waves.

Two Sources separated in Space with the Same Frequency

Suppose there are two sources of sinusoidal waves, each having the **same frequency** and in phase with each other. Let D be the separation of the two sources. The displacement at time t , and at a position located a distance r_1 away from the first source is:

$$d_1(r_1, t) = A(r_1)\sin(kr_1 - \omega t) \quad (76)$$

where the amplitude $A(r)$ will decrease with distance from the source. Similarly, the displacement due to the second source is

$$d_2(r_2, t) = A(r_2)\sin(kr_2 - \omega t) \quad (77)$$

where r_2 is the distance the observer is away from source 2.

If an observer receives both sources, the net displacement at the position of the observer is the sum of the displacements due to each source separately (Superposition Principle):

$$d_{net} = d_1 + d_2 \quad (78)$$

In general, this sum can be complicated. However, let's assume that $A(r_1)$ and $A(r_2)$ are approximately equal: $A(r_1) \approx A(r_2) \equiv A$. This is a good approximation if $D \ll r_i$. With this approximation,

$$d_{net} \approx A(\sin(kr_1 - \omega t) + \sin(kr_2 - \omega t)) \quad (79)$$

Now we can use our formula for adding two sin functions:

$$d_{net} \approx 2A\sin\left(\frac{k(r_1 + r_2)}{2} - \omega t\right)\cos\left(k\frac{r_1 - r_2}{2}\right) \quad (80)$$

Rearranging terms and defining $r_{ave} \equiv (r_1 + r_2)/2$ and $\Delta r \equiv (r_1 - r_2)$:

$$d_{net} \approx 2A\cos\left(k\frac{\Delta r}{2}\right)\sin(kr_{ave} - \omega t) \quad (81)$$

Since $k = 2\pi/\lambda$, we have

$$d_{net} \approx 2A\cos\left(\frac{\pi\Delta r}{\lambda}\right)\sin(kr_{ave} - \omega t) \quad (82)$$

The "sin" term oscillates in time and varies from +1 to -1. The factor $2A\cos(\pi\Delta r/\lambda)$ is the largest that d_{net} can become. It is the maximum amplitude of the net displacement:

$$A_{max} = 2A\cos\left(\frac{\pi\Delta r}{\lambda}\right) \quad (83)$$

This is an interesting result. $A_{max} = 2A$ if Δr equals $0, \pm\lambda, \pm2\lambda, \dots, \pm N\lambda$ where N is an integer. In this case, the two waves add "constructively", with the net amplitude being $2A$.

If, on the other hand, $\Delta r = \pm\lambda/2, \pm3\lambda/2, \dots, \pm(N + 1/2)\lambda$ then the **net displacement equals ZERO**. This is because $\cos(\pi/2) = 0$, etc. In this case, the two waves cancel each other.

The superposition principle leads to what is called "interference effects". If the two amplitudes add constructively, it is termed "constructive interference". If the two waves cancel out, it is called "destructive interference". The general expression for the interference amplitude is $A_{max} = 2A\cos(\pi\Delta r/\lambda)$. Another way of looking at the interference effect is to notice that the "phase" difference between d_1 and d_2 is given by $\phi = 2\pi\Delta r/\lambda$. If the extra distance Δr equals $\lambda/2$ then the waves are 180° out of phase and cancel out. If the extra distance Δr is a multiple of a wave length λ then the waves are "in phase" and add together.

We will do a number of examples and demo's in class which illustrate "two source" interference including: two sound sources and double slit interference.

Two sources of Sinusoidal waves with slightly Different Frequencies

Suppose there are two sources of sinusoidal waves located next to each other. One source has a frequency of f_1 , and the other a frequency of f_2 , with $f_1 \approx f_2$. The sources have the same amplitude A . If an observer "listens" to both sounds at once, the net displacement is:

$$d_{net}(t) = A\sin(2\pi f_1 t) + A\sin(2\pi f_2 t) \quad (84)$$

Using our formula for adding sin functions:

$$d_{net}(t) = 2A\cos(2\pi(\frac{f_1 - f_2}{2})t)\sin(2\pi(\frac{f_1 + f_2}{2})t) \quad (85)$$

Defining $\Delta f \equiv f_1 - f_2$ and $f_{ave} \equiv (f_1 + f_2)/2$, we have

$$d_{net}(t) = 2A\cos(\pi(\Delta f)t)\sin(2\pi f_{ave}t) \quad (86)$$

The sin term oscillates in time with a frequency between f_1 and f_2 . The cos term oscillates much slower, $\Delta f \ll f_{ave}$, since $f_1 \approx f_2$. So, the observer hears a sound with frequency $f_{ave} \approx f_1$, whose amplitude slowly changes in time according to $2A|\cos(\pi(\Delta f)t)|$. The result is a sound that gets loud, soft, loud, soft, etc. This

phenomena is called **beats**. The time T between loud sounds is $(\Delta f)T = 1$, or $1/T = \Delta f$. The frequency of the loud sounds, called the **beat frequency**, f_{beat} equals $1/T$. Thus, $f_{beat} = \Delta f$,

$$f_{beat} = |f_1 - f_2| \quad (87)$$

Standing Waves

If waves are formed in a medium that has boundaries, standing waves can form. Here we will consider the case of waves that can move in only one dimension. In one dimension, having boundaries means that the waves are confined to a specific length L of the medium. At a boundary, a wave is reflected back. Thus, there will be the superposition of waves traveling to the right and waves traveling to the left. Consider this happening for two sinusoidal waves of the same amplitude and frequency. The displacement for one wave will be:

$$d_1(x, t) = A \sin(kx + \omega t) \quad (88)$$

traveling to the left, and for the wave traveling to the right:

$$d_2(x, t) = A \sin(kx - \omega t) \quad (89)$$

The net displacement is the sum of these two waves (Superposition Principle) $d_{net} = d_1 + d_2$:

$$d_{net}(x, t) = A \sin(kx + \omega t) + A \sin(kx - \omega t) \quad (90)$$

Using the formula for adding two sin waves we obtain:

$$d_{net} = 2A \sin(kx) \cos(\omega t) \quad (91)$$

This is a simple result. The "x" part is separate from the "t" part. The wave doesn't travel anywhere, it is an equal sum of waves moving to the right and to the left. The sinusoidal shape $2A \sin(kx) = 2A \sin(2\pi x/\lambda)$ in space is multiplied by $\cos(\omega t)$. Thus, the shape "oscillates" in time with frequency $\omega/(2\pi)$. Places where $\sin(2\pi x/\lambda)$ are zero will not move at all. These positions are called **nodes**. Places where $\sin(2\pi x/\lambda)$ are ± 1 will have the largest amplitudes. These positions are called **anti-nodes**.

The distance between a node and an adjacent anti-node is one fourth of a wavelength: $\lambda/4$.

In this course we will consider cases in which the boundaries are either "closed" or "open". If the boundary is closed, there will be a node at the boundary. If the boundary is open, there will be an anti-node at the boundary. For a standing wave to exist in a region of length L , an appropriate number of quarter wavelengths ($\lambda/4$) must fit into the boundary, and the boundary conditions must be satisfied. In lecture we will examine examples of standing waves in pipes and on strings.

When we derived the standing wave equation above we added perfect sinusoidal functions. You might be wondering, is this the most general case. That is, what shapes can standing waves be? Must they always be perfect sinusoidal forms? We can answer this question by starting from the wave equation: $v^2 d^2 d'' / dx^2 = d^2 d'' / dt^2$. By stationary standing waves, we mean displacements that are separable in space and time. That is, $d(x, t) = f(x)g(t)$. We can substitute this form for d into our wave equation and determine what functions f and g are allowed:

$$v^2 \frac{d^2(f(x)g(t))}{dx^2} = \frac{d^2(f(x)g(t))}{dt^2} \quad (92)$$

we can factor out the $g(t)$ on the left side and the $f(x)$ on the right:

$$v^2 g(t) \frac{d^2 f(x)}{dx^2} = f(x) \frac{d^2 g(t)}{dt^2} \quad (93)$$

Dividing by $f(x)g(t)$ on both sides gives:

$$v^2 \frac{1}{f(x)} \frac{d^2 f(x)}{dx^2} = \frac{1}{g(t)} \frac{d^2 g(t)}{dt^2} \quad (94)$$

where v is a constant. The left side of the equation is only dependent on the variable x , and the right side of the equation is only dependent on the variable t . If this equation is to be true for all x and t , then each side must equal a constant (the same one) C . Thus:

$$v^2 \frac{1}{f(x)} \frac{d^2 f(x)}{dx^2} = C \quad (95)$$

For oscillatory solutions, the constant on the right must be less than zero, so

$$\frac{d^2 f(x)}{dx^2} = -\frac{|C|}{v^2} f(x) \quad (96)$$

Similarly, the equation for $g(t)$ is

$$\frac{d^2g(t)}{dt^2} = -|C|g(t) \quad (97)$$

The only function whose **second derivative is proportional to minus the function itself** is a **sinusoidal** function. Thus, standing waves in one dimensional media, whose wave equation are of the form above (linear media), will be perfect sinusoidal functions of position multiplied by perfect sinusoidal functions of time.

Summary

The past three weeks we have covered various topics on oscillations and waves. Although we defined various terms and looked at different phenomena, I think there are only three main "physics" concepts: 1) the linear restoring force and the motion it produces; 2) the one dimensional "wave equation"; and 3) the superposition principle for "waves".

1) When considering periodic motion in one dimension, we asked the following question: What kind of restoring force will produce perfect sinusoidal motion? We answered this question using Newton's second law, $F_{net} = ma$. We discovered that a force of the form: $F = -Cx$, or a linear restoring force, gives an acceleration a of $a = -(C/m)x$, or $d^2x/dt^2 = -(C/m)x$. The only function whose second derivative is equal to minus itself is a sinusoidal function. The most general solution can be written as: $x(t) = A\sin(\omega t + \phi)$, where $\omega = \sqrt{C/m}$. Not only did we find the special force for simple harmonic motion, we derived the formula which relates the period of the motion to the parameters of the force: $T = 2\pi/\omega = 2\pi\sqrt{m/C}$.

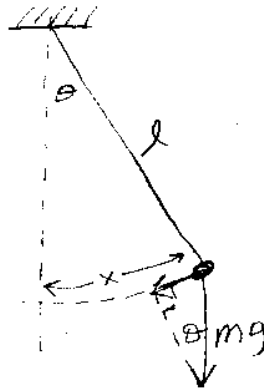
2) In the discussion on waves (i.e. disturbances of a medium) we observed that a one dimensional medium can support a traveling wave that moves to the right or left with a definite speed and a shape that doesn't change. We asked the question: why does this happen? We answered the question by applying Newton's law of motion to a string. The equation $a = F/m$ turned into a differential equation of the form: $d^2y/dt^2 = (F/\mu)d^2y/dx^2$. The acceleration became the second derivative of the displacement with respect to time, and the force became proportional to the second derivative of displacement with respect to position. We showed that the general solution to this "wave equation" was any function of the form $g(x - vt)$. Not only did we explain the wave motion, but we obtained an expression for the speed of the wave: $v = \sqrt{F/\mu}$.

3) The last principle of physics that we discussed was motivated by the question: What is the net displacement of a medium if there are more than one source of waves? The answer to this question lies in a very important property of nature: the Superposition Principle: the net displacement is the sum of the displacements from the individual sources. Although the superposition principle is always true, interesting effects occur when the sources are **coherent**. In our case we considered sinusoidal mechanical sources of waves, and by coherent we mean that the sources have the same frequency. The superposition principle led to interference effects of which constructive and destructive interference were special cases.

Of all the properties of nature that we have studied the past three weeks, interference effects are probably the most significant. When it was discovered that interference effects also occur without a medium, our ideas of nature radically changed. As shown in lecture, electrons and photons (light particles) exhibit interference phenomena. The area of physics that deals with these properties is called Quantum Mechanics, and has been developing since the early 1900's. In the quantum world of molecules, atoms, and nuclei, interference effects rule. The advancements of quantum physics has changed our view of nature, and led to the digital revolution. To learn more about quantum physics take our Phy235 lecture class.

$$F_{\text{restoring}} = -mg \sin \theta$$

$$= -mg \sin\left(\frac{x}{l}\right)$$



SIMPLE PENDULUM

