Exercises on Voltage, Capacitance and Circuits

Exercise 1.1
Instead of buying a capacitor, you decide to make one. Your capacitor consists of two circular metal plates, each with a radius of 5 cm. The plates are parallel to each other and separated by a distance of 1 mm. You connect a 9 volt battery across the plates. Find: the capacitance of the capacitor, the charge on each plate and the excess number of electrons on the negative plate.

You have made a simple parallel plate capacitor. The capacitance is equal to $C = \epsilon_0 A/d$:

$$\begin{aligned} C &= \epsilon_0 \frac{A}{d} \\ &= (8.85 \times 10^{-12}) \frac{\pi (0.05)^2}{0.001} \\ &= 6.95 \times 10^{-11}\ F \end{aligned}$$

To find the charge on each plate, we use the property of capacitance, $Q = CV$:

$$\begin{aligned} Q &= CV \\ Q &= (6.95 \times 10^{-11})(9) \\ Q &= 6.25 \times 10^{-10}\ C \end{aligned}$$

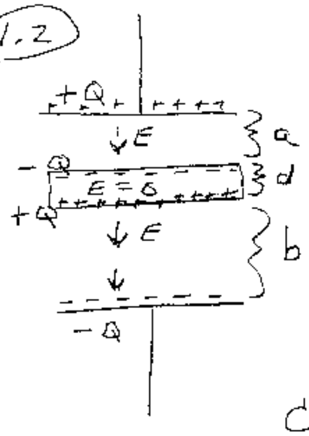
To find the number of electrons N on the negative plate, we use the fact that the charge on an electron is 1.6×10^{-19} Coulombs:

$$N = \frac{6.25 \times 10^{-10}}{1.6 \times 10^{-19}} \approx 3.91 \times 10^{9} \quad (1)$$

Even though the capacitance and net charge is small, it amounts to a transfer of nearly four billion electrons.

Exercise 1.2
You decide to make another capacitor as shown in the figure. You start with two parallel conducting plates, of area A. Then you place another conductor in the middle that is a thick plate of thickness d. The top of the thick plate is a distance a from

(1.2)



$$E = \frac{Q/A}{\epsilon_0}$$

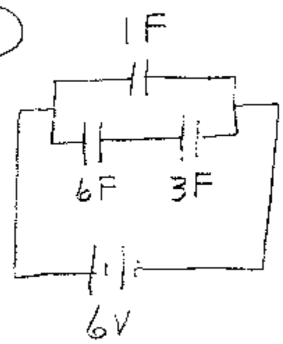
$$V = Ea + 0d + Eb$$

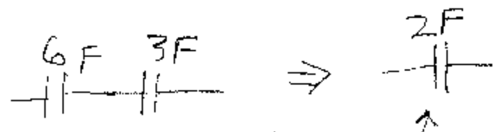
$$V = E(a+b)$$

$$V = \frac{Q}{A\epsilon_0}(a+b)$$

$$C = \frac{Q}{V} = \boxed{\frac{A\epsilon_0}{(a+b)}}$$

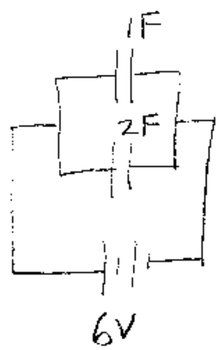
(1.3)





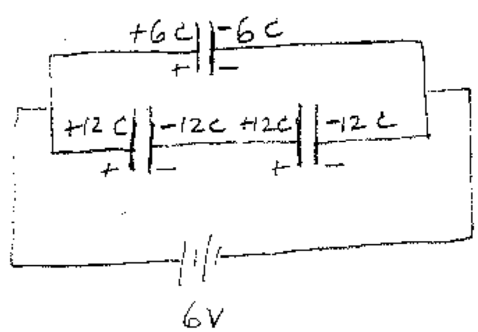
$$\frac{1}{C} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

$C = 2F$



$Q = CV = 1(6) = 6$ coulombs

$Q = CV = 2(6) = 12$ coulombs



the top plate and the bottom of the thick plate is a distance b from the bottom plate. What is the capacitance of your capacitor in terms of A, a, b, d, and ϵ_0?

To find the capacitance of a set of conductors, one can first put a charge of $+Q$ on one plate and a charge of $-Q$ on the other. Then determine the electric field between the conductors and integrate to find the voltage difference. Then $C = Q/V$. Place $+Q$ on the top plate and $-Q$ on the bottom plate. An electric field is produce inside the capacitor, however there is no electric field inside the thick conductor. In order to have $\vec{E} = 0$ inside the thick conductor, an amount of charge $-Q$ must collect on the top part of the thick conductor and $+Q$ on the bottom part.

Therefore, the electric field between the top plate and thick conductor is $E = \sigma/\epsilon_0 = Q/(A\epsilon_0)$, pointing downward. Similarly, the electric field between the bottom plate and the thick conductor is $E = \sigma/\epsilon_0 = Q/(A\epsilon_0)$ pointing downward. The potential difference between the top and bottom plates is therefore:

$$\begin{aligned} V &= \int \vec{E} \cdot \vec{dr} \\ &= Ea + Eb \\ &= \frac{\sigma}{\epsilon_0} a + \frac{\sigma}{\epsilon_0} b \\ &= \frac{Q}{A\epsilon} a + \frac{Q}{A\epsilon} b \\ &= \frac{Q}{A\epsilon_0}(a + b) \end{aligned}$$

Note that since there is no electric field inside the thick conductor the distance d doesn't enter into the expression for V. Since the capacitance equals Q/V we have

$$C = \frac{Q}{V} = \frac{A\epsilon_0}{a + b} \tag{2}$$

Exercise 1.3

You are given three capacitors of values, 1, 3, and 6 Farads. You connect them as shown in the figure. Find the total capacitance of the circuit and the charge on each capacitor.

We can add the "6" and "3" that are in series first, then add the result in parallel with the 1. Adding the series capacitors gives

$$\begin{aligned}\frac{1}{C_{36}} &= \frac{1}{3}+\frac{1}{6} \\ C_{36} &= 2\ Farads\end{aligned}$$

Adding this combination with the "1" in parallel yields

$$C_{total} = 2 + 1 = 3\ Farads \tag{3}$$

Since there is 6 volts across the "1" Farad capacitor, the plates have a net charge of

$$Q_1 = CV = 1(6) = 6\ Coulombs \tag{4}$$

The net charge on the plates of the series "3" and "6" Farad capacitors is

$$Q_{36} = CV = 2(6) = 12\ Coulombs \tag{5}$$

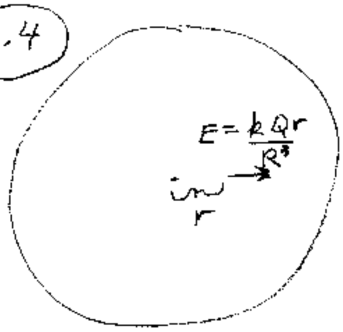
Note that the total charge supplied by the battery is $Q = CV = 3(6) = 18$ Coulombs, also equals $6 + 12$ Coulombs.

Exercise 1.4
Consider a uniformly charged sphere that has a total charge of Q that has a radius of R. What is the electrostatic potential difference, ΔV, between the center of the sphere and the surface of the sphere?

To find an expression for the potential difference between these two spots, we need to integrate the electric field from the center to the surface of the sphere: $\Delta V = \int \vec{E} \cdot \vec{dr}$. Integration is necessary in this case, since the electric field is not constant inside the solid sphere. To start, we first need to find an expression for the electric field inside the uniformly charged sphere. Because there is spherical symmetry, Gauss's Law will be useful for us.

Since the charge distribution has spherical symmetry, **the electric field vector must point radially away from the center of the sphere**, and **the magnitude of the electric field can only depend on r**, where r is the distance to the sphere's center. Thus, $\vec{E}(\vec{r}) = E(r)\hat{r}$, where $r = |\vec{r}|$. Because of this spherical symmetry, a useful surface to choose is a spherical shell. On the surface of the shell, the electric field is perpendicular to the surface and has the same magnitude at any point on the surface. The surface integral of $\vec{E} \cdot \vec{dA}$ is easy to calculate:

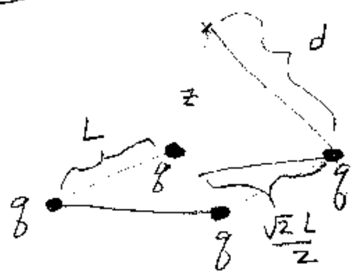
(1.4)



$$\Delta V = \int_0^R \frac{kQr}{R^3}\, dr$$

$$\Delta V = \frac{kQ}{2R}$$

(1.5)

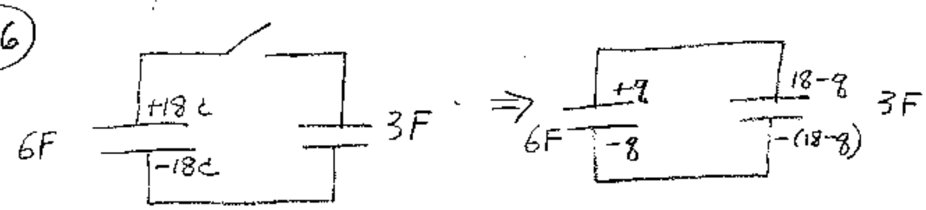


$$d^2 = z^2 + \left(\frac{\sqrt{2}L}{2}\right)^2$$

$$d = \sqrt{z^2 + L^2/2}$$

$$V = \frac{4kq}{\sqrt{z^2 + L^2/2}}$$

(1.6)



$$\frac{q}{6} = \frac{18-q}{3}$$

$q = 12$ coulombs

$18 - q = 6$ coulombs

$$
\begin{aligned}
\int\int \vec{E} \cdot d\vec{A} &= \frac{Q_{inside}}{\epsilon_0} \\
4\pi r^2 E &= \frac{Q_{inside}}{\epsilon_0}
\end{aligned}
$$

For points inside the sphere we choose our spherical shell to be inside the sphere, that is $r < R$. In this case, the amount of charge inside the shell is just the volume inside the shell times the charge density: $Q_{inside} = (4/3)\pi r^3 \rho$ where ρ is the charge density. Since the charge density of the sphere is $\rho = Q/((4/3)\pi R^3)$, we have

$$
Q_{inside} = Q\frac{r^3}{R^3} \tag{6}
$$

Substituting into the expression of Gauss's Law above yields

$$
\begin{aligned}
4\pi r^2 E &= \frac{Q_{inside}}{\epsilon_0} \\
4\pi r^2 E &= Q\frac{r^3}{R^3 \epsilon_0} \\
E &= \frac{Qr}{4\pi\epsilon_0 R^3} \\
E &= kQ\frac{r}{R^3}
\end{aligned}
$$

which is the magnitude of the electric field for points inside the uniformly charged sphere .

Now we can find the potential difference between the center and a point on the surface. We just need to integrate $\vec{E} \cdot d\vec{r}$ from the center to the surface of the sphere:

$$
\begin{aligned}
\Delta V &= \int \vec{E} \cdot d\vec{r} \\
&= \int_0^R kQ\frac{r}{R^3}\, dr \\
&= \frac{kQ}{2R}
\end{aligned}
$$

Exercise 1.5

Four small "point" objects are located at the corners of a square of side L. Each object has a charge q. Find an expression for the electric potential for locations on the axis that goes through the center of the square and perpendicular to its surface.

Let the z-axis go through the center of the square and perpendicular to the surface. We need to find $V(z)$. The electric potential a distance r away from a "point" charge is kq/r. We need to find the distance d away from a point on the z-axis to a corner of the square. From the figure, we can see that $d = \sqrt{z^2 + L^2/2}$. Thus, the electric potential at locations on the z-axis due to one of the charges is $V(z) = kq/\sqrt{z^2 + L^2/2}$. Since V is a scalar, we can just add the electric potential from the four charges:

$$V(z) = \frac{4kq}{\sqrt{z^2 + L^2/2}} \tag{7}$$

From this expression we can calculate the electric field in the "z" direction by differentiating $V(z)$ with respect to z:

$$\begin{aligned} E_z &= -\frac{\partial V}{\partial z} \\ &= \frac{4kqz}{(z^2 + L^2/2)^{3/2}} \end{aligned}$$

Note that this is the same expression we obtained in a homework problem by adding up the four electric field vectors. Taking the derivative of the potential is much easier.

Exercise 1.6
Consider the two capacitors connected as shown in the figure. The capacitor on the left is 6 Farads and the one on the right is 3 Farads. Initially the switch is open, and the capacitor on the left has ± 18 Coulombs on its plates. That is, it has a potential difference of 3 Volts. Initially, the capacitor on the right is uncharged. The switch is now closed. After the charges have settled down, determine the final charges on the two capacitors and the amount of electrostatic energy lost.

Initially, the electrostatic energy is is $U_{tot} = QV/2 = 18(3)/2 = 27$ Joules. After the switch is closed, the charge can move from one capacitor to the other. The charges will stop moving when the voltage across each capacitor is the same. Let q be the final charge on the 6 Farad capacitor. Since the total charge is 18 Coulombs, the charge on the 3 Farad capacitor will be $18 - q$. Since $V = Q/C$, we have

$$\begin{aligned}
\frac{q}{6} &= \frac{18-q}{3} \\
3q &= 6(18) - 6q \\
q &= 12\,Coulombs
\end{aligned}$$

Thus, the 6 Farad capacitor ends up with ± 12 Coulombs on the plates, and the 3 Farad capacitor ends up with ± 6 Coulombs. The common voltage across each capacitor is $V = Q/C = 12/6 = 6/3 = 2$ Volts.

To find the final energy, we just add the energies of the two capacitors:

$$\begin{aligned}
U_{final} &= \frac{Q_1 V_1}{2} + \frac{Q_2 V_2}{2} \\
&= \frac{12(2)}{2} + \frac{6(2)}{2} \\
&= 18\,Joules
\end{aligned}$$

Thus, there was a loss of $27 - 18 = 9$ Joules of electrostatic energy. The 9 Joules of energy were transfered into another form, a form that we will learn about later on in the course.

Exercise 1.7

How much electrostatic energy does a uniformly charged solid sphere of radius R have? That is, if we were to make a uniformly charged solid sphere, how much energy would it take to bring all the charges from infinity into the ball of charge? Let Q be the charge of the sphere.

This is a classic problem in electrostatics. The easiest way to solve for the electrostatic energy is to use the expression for the electrostatic energy density:

$$U = \frac{\epsilon_0 E^2}{2} \tag{8}$$

We need to know the electric field at all locations in space for a uniformly charged sphere. We have solved this problem before (Exercise 1.4 above). Let r be the distance from the center of the sphere. Then for $r < R$ we found that

$$|\vec{E}| = kQ\frac{r}{R^3} \tag{9}$$

and for values of $r > R$ we found in lecture that

$$|\vec{E}| = k\frac{Q}{r^2} \tag{10}$$

We now need to add up the contribution to the energy over all space. First inside the sphere:

$$\begin{aligned} U_{r<R} &= \int_0^R \frac{\epsilon_0 E^2}{2} dV \\ &= \frac{\epsilon_0}{2}\int_0^R \frac{k^2Q^2r^2}{R^6}\, 4\pi r^2 dr \\ &= \frac{kQ^2}{10R} \end{aligned}$$

Note that the volume element dV is equal to $4\pi r^2 dr$ since there is spherical symmetry and the integral is over r. For the space outside of the sphere ($R > R$), we have

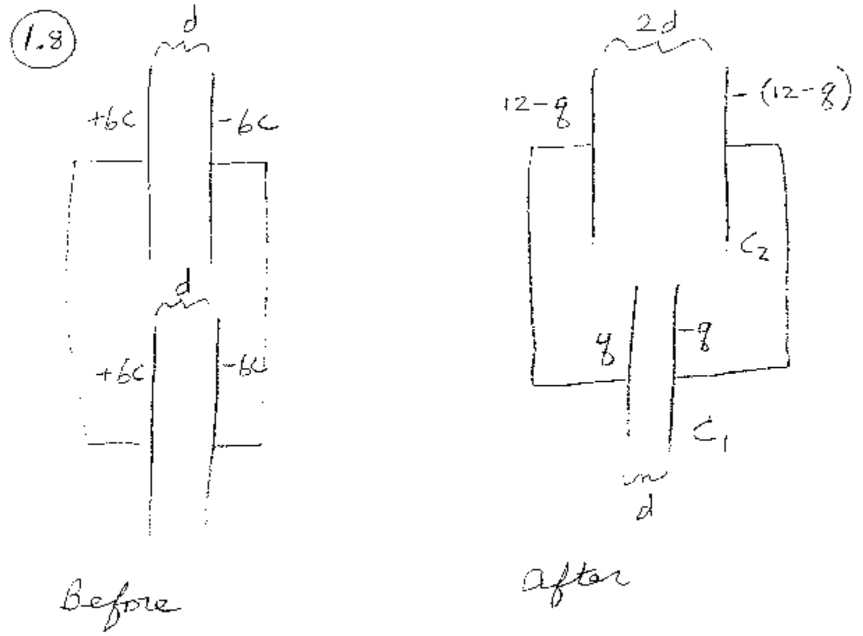
$$\begin{aligned} U_{r>R} &= \int_0^R \frac{\epsilon_0 E^2}{2} dV \\ &= \frac{\epsilon_0}{2}\int_R^\infty \frac{k^2Q^2}{r^4}\, 4\pi r^2 dr \\ &= \frac{kQ^2}{2R} \end{aligned}$$

Adding the two contributions to the electrostatic energy gives:

$$\begin{aligned} U_{tot} &= U_{r<R} + U_{r>R} \\ &= \frac{kQ^2}{10R} + \frac{kQ^2}{2R} \\ &= \frac{3}{5}\frac{kQ^2}{R} \end{aligned}$$

This expression represents the energy it would take to bring together a total charge equal to Q into a sphere. As $R \to 0$ this energy approaches infinity. One reason this is an interesting result is that the electron is essentially a "point" particle. We don't have an upper limit for the radius of an electron. Experiments show that the electron's radius is less than 10^{-19} meters. This small value for R would make the electrostatic

(1.8)



$$\frac{q}{C_1} = \frac{12-q}{C_2}$$

$$\frac{C_2}{C_1} = \frac{12-q}{q}$$

but $$\frac{C_2}{C_1} = \frac{\epsilon_0 A/2d}{\epsilon_0 A/d} = \frac{1}{2}$$

$$\frac{1}{2} = \frac{12-q}{q}$$

$$q = 24 - 2q \implies \begin{array}{l} q = 8 \text{ Coulombs} \\ 12 - q = 4 \text{ Coulombs} \end{array}$$

energy of the electron greater than its rest mass energy. This analysis demonstrates that Classical physics breaks down at very small length scales.

Exercise 1.8
Two identical parallel plate capacitors are connected in parallel as shown in the figure. Initially, each one has a charge of ± 6 Coulombs on the plates, which are separated by a distance d. Now, on one of the capacitors the plates are pulled apart until the separation between them is $2d$. What is the final charge distribution on the plates of the capacitors?

Here is the key "physics": since they are connected in parallel, the voltage across each capacitor is the same. Let q be the final charge on capacitor "1", whose plates have a final separation of d. Then $12 - q$ is the final charge on the other capacitor "2", whose plates have a final separation of $2d$. This is true, since the sum of the charge on both capacitors is still 12 Coulombs. Using $V = Q/C$, we have

$$\begin{aligned} \frac{q}{C_1} &= \frac{12-q}{C_2} \\ \frac{C_2}{C_1} &= \frac{12-q}{q} \end{aligned}$$

Since for a parallel plate capacitor, $C = \epsilon_0 A/d$, we have $C_2/C_1 = d/(2d) = 1/2$. That is, a larger separation between the plates results in a smaller capacitance. In this case, the ratio of capacitances is $1/2$ since the distance is twice as large in capacitor "2":

$$\begin{aligned} \frac{1}{2} &= \frac{12-q}{q} \\ q &= 8\ Coulombs \end{aligned}$$

The charge on capacitor "2" is therefore $(12 - q) = 4$ Coulombs. When the plates of capacitor "2" are separated, 2 Coulombs of charge flows from it to the other capacitor. We demonstrated this in lecture. When the plates of the "slide" capacitor were moved apart, charge went onto the electroscope and the leaves spread apart more.

Exercise 2.1
You have a wire that has a resistance of R_0. However, you would like a wire to have

a resistance that is twice as large, $2R_0$. Instead of buying a new wire, you decide to melt down the original wire and reform it to have a resistance of $2R_0$ using the same amount of material. What should the new length be?

We need to know how the resistance of a wire depends on its dimensions: $R = \rho l/A$, where ρ is the resistivity of the material, l is the length and A is the cross sectional area. When we make the new wire, the resistivity does not change since we are using the same material. We need to change l and A in such a way as to double the resistance. Since we are using the same amount of material the final volume will equal the original volume:

$$lA = l_0 A_0 \tag{11}$$

where l_0 and A_0 are the length and area of the original wire. The new area A will be equal to $A = l_0 A_0/l$. Since we want the resistance to be twice as large:

$$\begin{aligned} 2R_0 &= \frac{\rho l}{A} \\ 2R_0 &= \frac{\rho l^2}{l_0 A_0} \end{aligned}$$

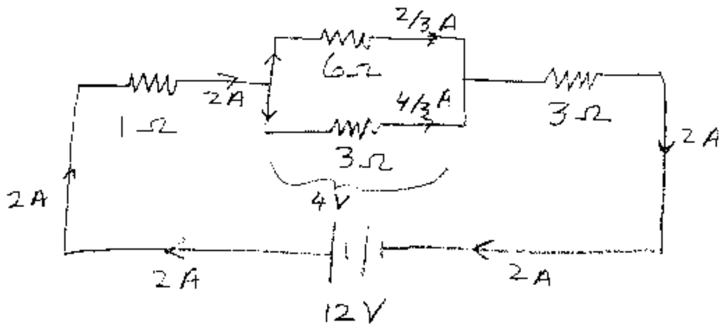
For the original wire, $R_0 = \rho l_0/A_0$. With this substitution, we have:

$$\begin{aligned} 2\frac{\rho l_0}{A_0} &= \frac{\rho l^2}{l_0 A_0} \\ 2l_0 &= \frac{l^2}{l_0} \\ l &= \sqrt{2}\, l_0 \end{aligned}$$

So the new wire should have a length that is $\sqrt{2}$ greater than the original wire, and an area that is $\sqrt{2}$ smaller than the original wire. This way, the volume is the same, but the resistance is doubled.

Exercise 2.2
Consider the circuit shown in the figure in which a 6Ω resistor and a 3Ω resister are in parallel. This parallel combination is in series with a 1Ω and a 3Ω resistor. Find

2.2



$$\frac{1}{R_{36}} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2} \qquad R_{36} = 2\Omega$$

$$I_{TOT} = \frac{12}{6} = 2A$$

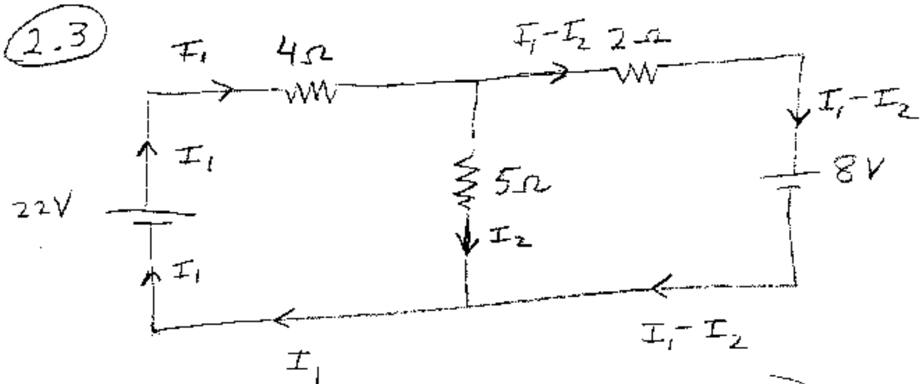
$$R_{TOT} = 1 + 2 + 3 = 6\Omega$$

$$I_6 = \frac{4}{6} = \frac{2}{3}A$$

$$V_{36} = 12 - 2 - 6 = 4V$$

$$I_3 = 2 - \frac{2}{3} = \frac{4}{3}A$$

2.3



$$\begin{cases} +22 - 4I_1 - 5I_2 = 0 \\ +22 - 4I_1 - 2(I_1 - I_2) - 8 = 0 \end{cases}$$

$\rightarrow$ $I_1 = 3$ Amps $\quad I_2 = 2$ Amps the current through and the voltage across each resistor.

This combination of resistors can be broken up into parallel and series sections. This is nice, because one can calculate the equivalent resistance of the whole circuit and find the current delivered by the battery. Once this is done, one can go piece by piece to determine the currents and voltages across the resistors. First we determine the resistance of the whole circuit.

The 6Ω and 3Ω resistors in parallel are equivalent to

$$\begin{aligned} \frac{1}{R_{36}} &= \frac{1}{3} + \frac{1}{6} \\ R_{36} &= 2\Omega \end{aligned}$$

2 Ohms. This 2Ω parallel combination is in series with the other two resistors, so the total resistance of the circuit, R_T, is

$$R_T = 1 + 2 + 3 = 6\Omega \tag{12}$$

The current that is supplied by the battery is $I = V/R_T$, or

$$I_{battery} = \frac{12}{6} = 2 \; amps \tag{13}$$

Thus, 2 amps flows through the 1Ω resistor and the 3Ω resistor that is in series. Now the voltages across all the resistors can be determined. Across the 1Ω resistor, $V = IR = 2(1) = 2$ Volts. Across the 3Ω resistor in series with the battery, $V = IR = 2(3) = 6$ Volts. The voltage across the 3Ω and 6Ω resistors in parallel are $V = 12 - 2 - 6 = 4$ volts. The current through the 6Ω resistor is therefore $I = V/R = 4/6 = 2/3$ Amps. The current through the 3Ω resistor (in the parallel combination) is $I = V/R = 4/3$ Amps.

Exercise 2.3

Consider the combination of resistors and batteries shown in the figure. Find the current through and the voltage across each resistor.

In this case it is not possible to combine the resistors into equivalent series and parallel pieces. We cannot determine a total resistance for the circuit as we did in the last example. We need to use the basic physics equations: Kirchoff's Laws. We start by assigning a value for the current in the various wires. Let I_1 be the current that flows our of the 22 volt battery, and also through the 4Ω resistor. Let I_2 be the current through the 5Ω resistor. Since the charge flow into a junction equals the charge flow out, the current through the 2Ω resistor will be $(I_1 - I_2)$. This is also the current that flows into the "+" side of the 8 volt battery. Using this labeling of the current we have satisfied Kirchoff's current law.

Now we need to express Kirchoff's voltage law mathematically. If the currents are not changing in time (and there are no changing magnetic fields) The voltage changes around a closed loop equals zero. Starting in the lower left corner and going clockwise around the left loop yields:

$$\begin{aligned} 22 - 4I_1 - 5I_2 &= 0 \\ 22 &= 4I_1 + 5I_2 \end{aligned}$$

Starting in the lower left corner and going clockwise around the outer loop yields:

$$\begin{aligned} 22 - 4I_1 - 2(I_1 - I_2) - 8 &= 0 \\ 14 &= 6I_1 - 2I_2 \end{aligned}$$

Now we need to solve the two equations for the two unknowns, I_1 and I_2. Multiplying the top one by 6 and the bottom one by 4 and subtracting allows us to solve for I_2: $I_2 = 2$ Amps. Substituting $I_2 = 2$ into one of the equations results in $I_1 = 3$ Amps.

Thus, 3 amps flows out of (and back into) the 22 volt battery as well as through the 4Ω resistor. A current of 2 amps flows through the 5Ω resistor. A current of $(3 - 2) = 1$ amp flows through the 2Ω resistor. A current of 1 amp flows into the positive terminal of the 8 volt battery. That is, the 8 volt battery gets "charged-up". Note that we would have obtained the same results if we would have chosen diffent variables for the currents and used different paths for the voltage changes.

Exercise 2.4

Your flashlight contains a D-cell battery connected to a $.75\Omega$ light. The D-cell battery is 1.5 volts and has a capacity rating of 10 Amp-Hours. Find: the current flow in the circuit, the power dissipated by the light, the total energy that the battery originally had, and the length of time that the battery can continuously light the light before it is "dead".

a) The current is found using $I = V/R$:

$$I = \frac{V}{R} = \frac{1.5}{.75} = 2\ amps \tag{14}$$

b) The power dissipated by the light bulb is $P = I^2R$:

$$P = I^2R = (2)^2(.75) = 3\ Watts \tag{15}$$

or 3 Joules/sec.

c) The total energy originally available from the battery can be found by first calculating the available charge: 10 Amp-hours = (10 C/sec)(3600 sec) = 36000 Coulombs. Since this amount of charge flows through a potential voltage differenct of 1.5 volts, we have

$$Energy = (36000\ Coulombs)(1.5\ Volts) = 54000\ Joules \tag{16}$$

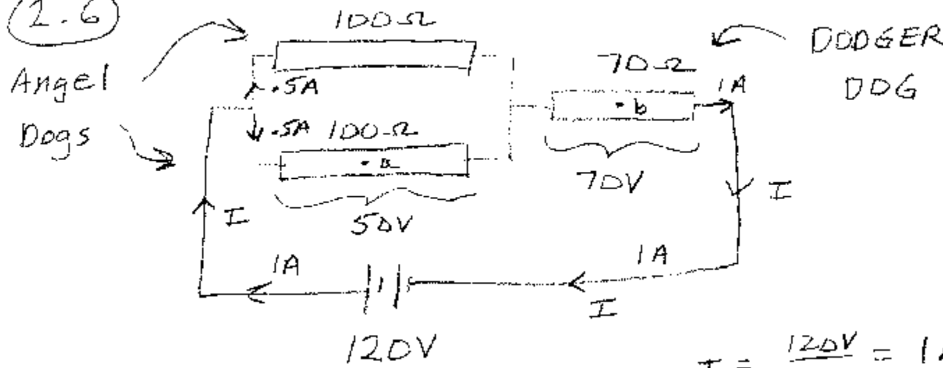
d) Since the battery uses up energy at a rate of 3 $Watts$ = 3 Joules/sec, it can last for a time of 54000/3 = 18000 seconds or 5 hours.

Exercise 2.5
You want to watch a TV show for four hours, but your parents think this will run the electric bill too high and cost too much money. Your TV has a power rating of 200 Watts. How much will it cost to watch the 4 hour show if electric energy costs 14 cents per KwHr?

To find the total cost of watching the TV show, we will first determine the amount of energy used in units of Kw-Hrs. Energy = $(.200\ Kw)(4\ hours)$ = 0.8 KwHr. This much energy will cost $(14\ cents/KwHr)(0.8\ KwHr)$ = 11.2 cents. Once you show your parents how little it costs, they will probably let you watch the show.

Exercise 2.6
You go to your friend's house for dinner and he is cooking hot dogs using an electrical circuit. The circuit is shown in the figure, and consist of 3 hot dogs. Two of the hot dogs, which are Angel Dogs, are connected in parallel, and this parallel combination is connected in series with one hot dog, which is a Dodger Dog. Your friend connects a voltage of 120 volts across the hot dog circuit. How much current flows through each hot dog, how much power is delivered to each hot dog, and what is the potential difference between the points a and b as shown in the figure. Point a is in the middle

(2.6)



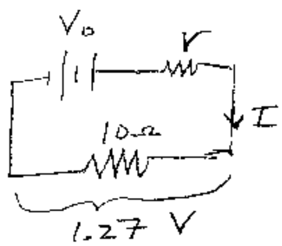
$$I = \frac{120V}{120\Omega} = 1\ \text{Amp}$$

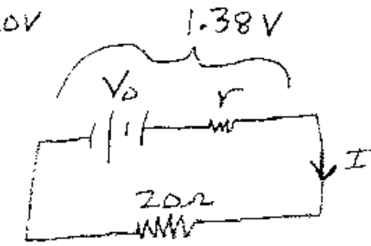
$$R_{TOT} = 50 + 70 = 120\ \Omega$$

$$P_{\text{DODGER DOG}} = 70(1)^2 = 70\ W \qquad P_{\text{ANGEL DOG}} = 100(.5)^2 = 25W$$

$$V_a - V_b = 25 + 35 = 60V$$

(2.7)





$$I = \frac{V_0}{10 + r} \qquad I = \frac{V_0}{20 + r}$$

$$1.27 = V_0 - rI \qquad 1.38 = V_0 - rI$$

$$1.27 = V_0 - \frac{rV_0}{10 + r} \qquad 1.38 = V_0 - \frac{rV_0}{20 + r}$$

$$\boxed{12.7 + 1.27r = 10V_0} \qquad \boxed{27.6 + 1.38r = 20V_0}$$

$$r \approx 1.9\Omega$$

$$V_0 \approx 1.51V$$

of one of the Angel Dogs, and point b is in the middle of the Dodger Dog. Each Angel Dog (in the parallel combination) has a resistance of 100Ω, and the Dodger Dog has a resistance of 70Ω.

Since the hot dogs are arranged in a simple parallel-series combination, we can calculate the total resistance of the circuit. The two Angel Dogs in parallel are equivalent to a resistance R of

$$\begin{aligned}\frac{1}{R} &= \frac{1}{100} + \frac{1}{100} \\ R &= 50\ \Omega\end{aligned}$$

The 50Ω Angel Dog combination is in series with the 70Ω Dodger Dog, so the total resistance of the hot dog circuit is $R_{tot} = 120\Omega$. The current that the battery delivers is therefore:

$$I = \frac{V}{R_{tot}} = \frac{120}{120} = 1\ Amp \tag{17}$$

This 1 Amp current goes through the Dodger Dog. The current splits up in the Angel Dog combination, so each Angel Dog has 0.5 Amp of current through it. Since $V = IR$, there is a 70 volt drop across the Dodger Dog, and a 50 volt drop across each Angel Dog. The power dissipated in a resistor is $P = i^2R$. Therefore, the power dissipated in the Dodger Dog is

$$P = 1^2 70 = 70\ Watts \tag{18}$$

The power dissipated in each of the Angel Dogs is

$$P = (0.5)^2 100 = 25\ Watts \tag{19}$$

Thus, the Dodger Dog will cook 70/25 times faster than the two Angel Dogs. I'd eat the Dodger Dog first.

The voltage drop from the middle of an Angel Dog to its end is $50/2 = 25$ volts. The voltage drop from the middle of the Dodger Dog to its end is $70/2 = 35$ volts. Thus, the voltage difference between points a and b is

$$\Delta V = \frac{50}{2} + \frac{70}{2} = 60\ volts \tag{20}$$

Exercise 2.7

A battery will have some internal resistance. We can model a real battery as a pure voltage source of voltage V_0 with a small resistor of resistance r in series with it. To determine V_0 and r you place a a resistor across the battery and measure its voltage. Here is the data you get: For a 10Ω resistor across the battery, the voltage is 1.27 volts, and for a 20Ω resistor across the battery, the voltage is 1.38 volts. What is V_0 and r for the battery?

The circuit is shown in the figure. When a 10Ω resistor is placed across the battery, it is in series with r. The current is therefore: $I = V_0/(10 + r)$. The voltage drop across the real battery is $1.27 = V_0 - Ir$. Substituting in for I gives

$$\begin{aligned} 1.27 &= V_0 - \frac{V_0 r}{10 + r} \\ 1.27 &= \frac{10V_0}{10 + r} \\ 12.7 + 1.27r &= 10V_0 \end{aligned}$$

A similar analysis can be when the 20Ω is connected. The current that flows is $I = V_0/(20 + r)$. The voltage drop across the real battery now is $1.38 = V_0 - Ir$. Substituting in for I gives

$$\begin{aligned} 1.38 &= V_0 - \frac{V_0 r}{20 + r} \\ 1.38 &= \frac{20V_0}{20 + r} \\ 27.6 + 1.38r &= 20V_0 \end{aligned}$$

We have two equations with two unknowns: V_0 and r. Solving for V_0 and r gives $r \approx 1.9\Omega$ and $V_0 \approx 1.51$ Volts.

Exercise 2.8

Your car battery has gone dead and you want to replace it with a large capacitor. Your old car battery is 12 volts and has a capacity of 40 amp-hours. What should the capacitance of the capacitor be so that a 12 volt difference across the plates will result in the same energy as the battery.

To solve the problem, we can first calculate the energy contained in the car battery. 40 amp-hours is equal to a charge of 40 $Coulombs/sec$ (3600 sec) = 144000 Coulombs.

The energy that a new battery has is $U = 12\ Volts(144000\ C) = 1.728 \times 10^6$ Joules. Equating this energy with the energy contained in a capacitor yields

$$
\begin{aligned}
U &= \frac{CV^2}{2} \\
1.728 \times 10^6 &= \frac{C\ 12^2}{2} \\
C &= 24000\ Farads
\end{aligned}
$$

This is a huge capacitance, and is impractical as a substitute for the battery. We could put a higher voltage on the capacitor, say 100 volts. Then the capacitance would be

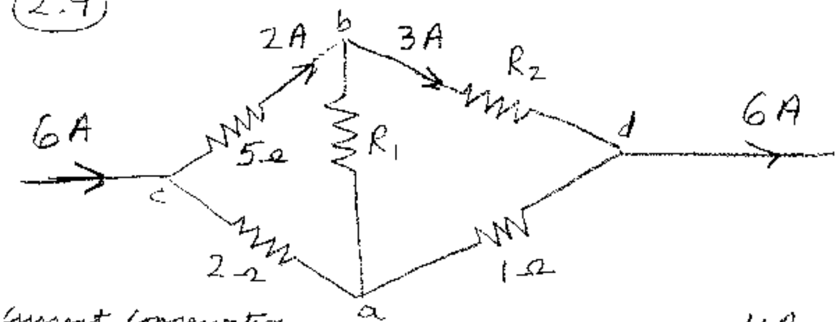
$$
\begin{aligned}
U &= \frac{CV^2}{2} \\
1.728 \times 10^6 &= \frac{C\ 100^2}{2} \\
C &= 245.6\ Farads
\end{aligned}
$$

This is still quite large. Until someone comes up with such a large capacitor, we had better just buy a new battery.

Exercise 2.9
Someone has connected 5 resistors as shown in the figure. This kind of connection is called a Wheatstone Bridge. If 6 Amps enters from the left, 2 Amps flow through the 5Ω resistor and 3 Amps flows through resistor R_2. Find the value of R_1, R_2, and the total resistance of the Wheatstone Bridge.

Since 2 Amps flows through the 5Ω resistor, there must be $6-2=4$ Amps flowing through the 2Ω resistor. Also, since there is 3 Amps flowing through resistor R_2 and 6 Amps leaves the circuit, the current that flows through the 1Ω resistor must be $6-3=3$ Amps. Therefore, 1 Amp must flow through R_1 from a to b.
The voltage at c, V_c, minus the voltage at a, V_a, is $V_c - V_a = 4(2) = 8$ Volts. Similarly $V_c - V_b = 5(2) = 10$ Volts. Therefore, the voltage drop across R_1 is $V_a - V_b = 10-8 = 2$ Volts. So the value of R_1 is $R_1 = 2/1 = 2\Omega$, since there is one amp flowing through R_1. The voltage difference between c and d is the same if one goes via the bottom path or the top path. So,

(2.9)



Current Conservation

The current from $c \to a$ = $6-2 = 4A$

The current from $a \to d$ = $6-3 = 3A$

The current from $a \to b$ = $3-2 = 1A$

$$V_c - V_a = I_{c\to a}(2\Omega) = 4(2) = 8V$$

$$V_c - V_b = I_{c\to b}(5\Omega) = 2(5) = 10V$$

$$\Rightarrow V_a - V_b = 2V$$

$$R_1 = \frac{V_a - V_b}{I_{a\to b}} = \frac{2V}{1A} = \boxed{2\Omega = R_1}$$

$$V_{c\to a\to d} = V_{c\to b\to d}$$

$$4(2) + 3(1) = 2(5) + 3R_2$$

$$\boxed{R_2 = \frac{1}{3}\Omega}$$

$$R_{TOTAL} = \frac{V_c - V_d}{6A} = \frac{8+3}{6} = \boxed{\frac{11}{6}\Omega}$$

$$
\begin{aligned}
4(2)+3(1) &= 2(5)+3R_2 \\
11 &= 10+3R_2 \\
R_2 &= \frac{1}{3}\,\Omega
\end{aligned}
$$

The voltage difference across the whole circuit is $V_c - V_d = 4(2) + 3(1) = 11$ Volts. Since 6 Amps flows through the circuit, the total Resistance is $11/6 \approx 1.83\Omega$.
Note: It is not possible to break up this circuit into a combination of series and parallel pieces. One has to use Kirchhoff's Laws to solve for the currents and voltage differences in this Wheatstone Bridge set-up.

Exercise 2.10
In your house there are three different wire gauges that are used, each with a different diameter and each with a different maximum allowed current. 14 gauge wire needs a 15 Amp circuit breaker, 12 gauge wire requires a 20 Amp circuit breaker, and 10 gauge wire needs a 30 Amp circuit breaker. The diameter and maximum allowed current are summarized in the table below:

Gauge	Diameter ($\times 10^{-3}$ in)	Maximum Current (A)
14	64	15
12	81	20
10	102	30

Determine the maximum allowed current density for each wire in units of Amps/cm^2. Which wire can have the highest current density?

In a wire, the current density equals $J = I/A$, so we can easily calculate the maximum allowed current density. The area of a 14 gauge wire is $A = \pi(32 \times 10^{-3})^2(2.54^2) \approx 0.0207\ cm^2$ since $1\ in \approx 2.54\ cm$. So

$$J_{14} = \frac{15\ A}{0.0207 cm^2} \approx 723 A/cm^2 \tag{21}$$

Similarly, the area of a 12 gauge wire is $A = \pi(40.5 \times 10^{-3})^2(2.54^2) \approx 0.0332\ cm^2$. So

$$J_{12} = \frac{20\ A}{0.0332 cm^2} \approx 602 A/cm^2 \tag{22}$$

For a 10 gauge wire, the area is $A = \pi(51 \times 10^{-3})^2(2.54)^2 \approx 0.0527\ cm^2$. So

$$J_{10} = \frac{30\ A}{0.0527cm^2} \approx 569A/cm^2 \tag{23}$$

We see that the 14 gauge wire is allowed to have a higher current density than the 12 or the 10 gauge wire. Usually 14 gauge wire is only connected to ceiling lights, and draws little current.

Exercise 2.11

You want to make a coaxial cylindrical resister, which is a conducting cylindrical material with a hole bored out in the middle along the axis. The inner radius is a, the outer radius is b, and the length of the cylindrical material is l. The conducting material has an electrical resistivity of ρ_e. If current is to flow radially from the inner radius to the outer radius, what is the resistance of your resistor? Express your answer in terms of l, a, b, and ρ_e.

In general, for a circuit element the resistance is defined as the ratio of the voltage across the element divided by the current through it. For a resistive material, the electric field inside the material is the common quantity for both the voltage difference and the current flow:

$$\begin{aligned} V &= \int_a^b \vec{E} \cdot \vec{dr} \\ \vec{J} &= \frac{\vec{E}}{\rho_e} \end{aligned}$$

Due to the cylindrical symmetry of the material, the electric field inside must point in the radial direction. In addition, the magnitude of the electric field can only depend on r the distance from the axis. For this geometry, we have used Gauss's Law to show that the magnitude of the electric field falls off as $1/r$ is the cylinder is infinite in length. Here we will assume that $l >> b$, so we can make the approximation that the cylinder can be considered as infinite. Thus, the electric field within the material must have the following r dependence:

$$E = \frac{c}{r} \tag{24}$$

where c is a constant. Since we know the electric field, we can calculate V and I in terms of c and the geometric constants. The voltage V is

$$
\begin{aligned}
V &= \int_a^b \vec{E} \cdot \vec{dr} \\
&= \int_a^b \frac{c}{r} dr \\
V &= c\, ln(b/a)
\end{aligned}
$$

To find the current, we need to integrate the current density over the area through which the current flows. The current is flowing through a cylindrical surface of area $2\pi rl$ at the radius r. Since $\vec{J} = \vec{E}/\rho_e$, we have

$$
\begin{aligned}
I &= \int \vec{J} \cdot \vec{dr} \\
&= J(r)\ (2\pi rl) \\
&= \frac{E}{\rho_e}\ 2\pi rl \\
&= \frac{c}{r\rho_e}\ 2\pi rl \\
&= \frac{c2\pi l}{\rho_e}
\end{aligned}
$$

for any coaxial cylindrical surface of radius r. The resistance can be found by dividing V by I:

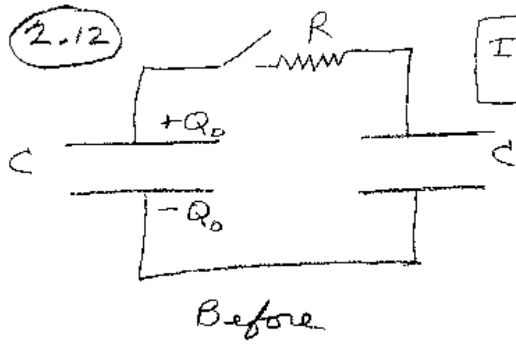
$$
\begin{aligned}
R &= \frac{V}{I} \\
&= \frac{c\, ln(b/a)}{c2\pi l/\rho_e} \\
R &= \frac{\rho_e}{2\pi l}\ ln(b/a)
\end{aligned}
$$

Note that the resistance has the correct units of resistivity times length.

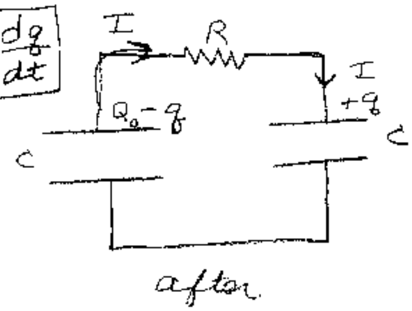
Exercise 2.12

Two identical capacitors are connected to a resistor as shown in the figure. Let the capacitors have a capacitance C and the resistor a resistance R. Initially the switch is open and the charge on the plates of the left capacitor is Q_0. At time $t = 0$ the switch

2.12



$$I = \frac{dq}{dt}$$



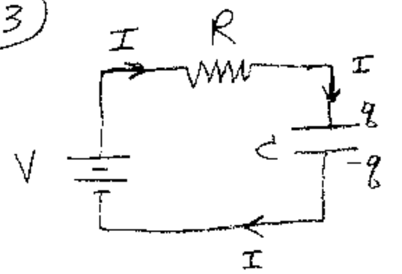
$$\frac{Q_0 - q}{C} - IR - \frac{q}{C} = 0$$

$$Q_0 - q - \frac{dq}{dt} CR - q = 0$$

$$Q_0 - 2q = RC\frac{dq}{dt}$$

$$q = \frac{Q_0}{2}\left(1 - e^{-2t/RC}\right)$$

2.13



$$V - IR - \frac{q}{C} = 0$$

$$I = \frac{dq}{dt}$$

$$V - R\frac{dq}{dt} - \frac{q}{C} = 0$$

$$CV - RC\frac{dq}{dt} - q = 0$$

$$CV - q = RC\frac{dq}{dt}$$

$$\int \frac{dt}{RC} = \int \frac{dq}{CV - q}$$

$$q = CV\left(1 - e^{-t/RC}\right)$$

$$I = \frac{dq}{dt}$$

$$I = \frac{V}{R} e^{-t/RC}$$

is closed. Find an expression for the charge on the right capacitor as a function of time.

Let $q(t)$ be the charge on the plates of the capacitor on the right at time t. Then the charge on the plates of the capacitor on the left is $Q_0 - q(t)$, since the total charge must add up to Q_0. Let the current at any time be $I(t)$. Then, adding up the voltage changes around the loop gives:

$$\frac{Q_0 - q}{C} - IR - \frac{q}{C} = 0 \tag{25}$$

Here we have used the properties that the voltage change across a capacitor is Q/C and across a resistor as IR. Note that the zero on the right is only true in the case of steady state currents. Here the current is changing, but the change is slow enough that the sum of the voltage drops around a closed loop is essentially zero. We will learn in the next section the reason that this approximation is good: the self inductance of the circuit is very small.

We now need to relate I to q. As seen in the figure, $I = +dq/dt$. We have a + sign here since we have chosen the direction of I to go into the + side of the right capacitor. Substituting into the above equation gives:

$$\begin{aligned} \frac{Q_0 - q}{C} - \frac{dq}{dt}R - \frac{q}{C} &= 0 \\ RC\frac{dq}{dt} &= Q_0 - 2q \end{aligned}$$

We can solve the differential equation above by integration:

$$\begin{aligned} \frac{dt}{RC} &= \frac{dq}{Q_0 - 2q} \\ \int \frac{dt}{RC} &= \int \frac{dq}{Q_0 - 2q} \\ \frac{t}{RC} &= -\frac{1}{2}\, ln(Q_0 - 2q)|_0^q \\ -\frac{2t}{RC} &= ln(Q_0 - 2q) - ln(Q_0) \\ e^{-2t/(RC)} &= \frac{Q_0 - 2q}{Q_0} \\ q &= \frac{Q_0}{2}(1 - e^{-2t/(RC)}) \end{aligned}$$

Initially, at $t = 0$, $q = 0$. As $t \to \infty$, then $q \to Q_0/2$. Note also that the capacitance in the RC time constant is $C/2$. Thus, in this circuit, it is as if the two capacitors are connected in series.

Exercise 2.13
Consider a resistor and capacitor connected in series to a battery as shown in the figure. Let the resistor have resistance R, the capacitor a capacitance C and the battery a voltage of V. Initially the switch is open and the capacitor is uncharged. When the switch is closed, current flows, and the capacitor charges up until it is reaches a final charge of $Q = CV$. After the capacitor has charged up, show that the energy supplied by the battery equals the energy dissipated by the resistor plus the final energy stored in the capacitor.

We label the charge on the plates of the capacitor as q and the current as I flowing from the positive side of the battery. Summing up the voltage changes around the loop:

$$V - RI - \frac{q}{c} = 0 \tag{26}$$

Using the direction we have defined for I, we have $I = dq/dt$. Substituting into the above equation yields:

$$R\frac{dq}{dt} = V - \frac{q}{C} \tag{27}$$

We can solve this differential equation as we did with the last problem. After multiplying by C and arranging terms we have:

$$\begin{aligned}
\int \frac{dt}{RC} &= \int \frac{dq}{CV - q} \\
\frac{t}{RC} &= -ln(CV - q)|_0^q \\
-\frac{t}{RC} &= ln(\frac{CV - q}{CV}) \\
q &= CV(1 - e^{-t/(RC)})
\end{aligned}$$

Note that the charge on the capacitor at $t = 0$ is zero, and as $t \to \infty$ the charge approaches $q(final) \to CV$. The current in the circuit is

$$
\begin{aligned}
I &= \frac{dq}{dt} \\
I &= \frac{V}{R} e^{-t/(RC)}
\end{aligned}
$$

Now we can calculated the energy transfer in the different elements in charging up the capacitor. The power supplied by the battery is $P = VI$. The energy supplied by the battery is found by integrating the power from $t = 0$ to $t = \infty$, $U_{battery} = \int P dt$:

$$
\begin{aligned}
U_{battery} &= \int_0^\infty VI \, dt \\
&= \int_0^\infty \frac{V^2}{R} e^{-t/(RC)} \, dt \\
&= V^2 C
\end{aligned}
$$

The power dissipated by the resistor is $P = IR = I^2 R$. Integrating the power from $t = 0$ to $t = \infty$ gives

$$
\begin{aligned}
U_{resistor} &= \int_0^\infty I^2 R \, dt \\
&= \int_0^\infty \frac{V^2}{R} e^{-2t/(RC)} \, dt \\
&= \frac{V^2 C}{2}
\end{aligned}
$$

Thus, the energy that was delivered to the capacitor is $U_{capacitor} = V^2 C - V^2 C/2 = V^2 C/2$, which is correct expression for the energy stored in a capacitor. All the energy is accounted for.