<div align="center">**Exercises on Electrostatics**</div>

Exercise 1.1
Suppose you have two small "point" objects separated by a distance of 1 cm. Each object has a diameter of 1×10^{-3} cm. One object has an excess of 3×10^{10} electrons and the other has an excess of 2×10^{10} electrons on it. What is the electrostatic force that they exert on each other?

Since the object's diameters are much less than their separation, $1 \times 10^{-3} << 1$, we can treat them as point objects and use Coulomb's law:

$$|F| = k\frac{q_1 q_2}{r^2} \tag{1}$$

where q_1 and q_2 is the net charge on each object. Here we have $q_1 = 3 \times 10^{10}$ $(1.6 \times 10^{-19}) = 4.8 \times 10^{-9}$ Coulombs, and $q_2 = 2 \times 10^{10}$ $(1.6 \times 10^{-19}) = 3.2 \times 10^{-9}$ Coulombs. So the force they each experience is

$$
\begin{aligned}
|F| &= (9 \times 10^9)\frac{(4.8 \times 10^{-9})(3.2 \times 10^{-9})}{0.01^2} \\
&\approx 1.38 \times 10^{-3} \; Newtons
\end{aligned}
$$

The objects repel each other with a force of 1.38×10^{-3} Newtons.

Exercise 1.2
Consider the 3 "point" objects shown in the figure. The object at the origin has a net charge of $+2q$ where q is some amount of positive charge. An object is located on the y-axis at $(0, a)$ that has a net charge of $-q$, where a is some unit of distance. A third object is located on the x-axis at $(2a, 0)$ that has a net charge of $3q$.

a) If the three charges are fixed in the configuration shown in the figure, what is the net electrostatic force that the object at the origin feels due to the other two? Express your answer in terms of q, a and k.

Since the objects are "point" objects, we can use **Coulomb's law** to find the individual forces. Then we can use the **superposition principle** to find the net force. This is the standard method for most electrostatic calculations. First the force on the object at the origin due to the object on the x-axis. The objects repel each other (since they have the same charge) with a force equal to:
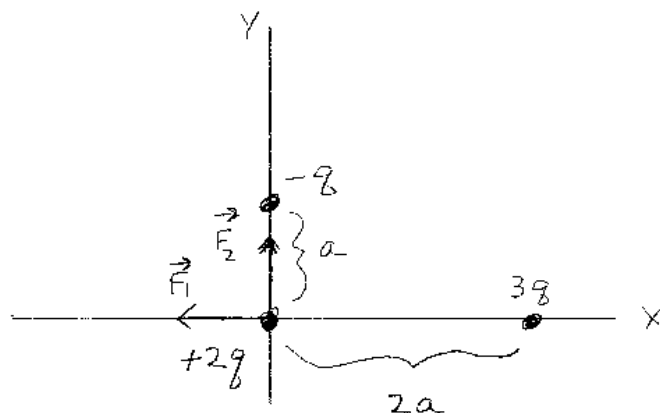
$$\left|\vec{F_1}\right| = \frac{k\,|2q|\,|3q|}{(2a)^2} = \frac{6}{4}\frac{kq^2}{a^2} = 1.5\frac{kq^2}{a^2}$$
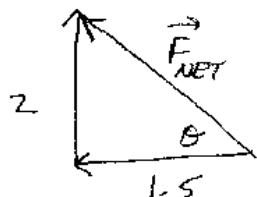
repulsive

$$\vec{F_1} = 1.5\frac{kq^2}{a^2}\,(-\hat{\imath})$$

$$\left|\vec{F_2}\right| = \frac{k\,|2q|\,|-q|}{a^2} = 2\frac{kq^2}{a^2} \qquad \text{attractive}$$

$$\vec{F_2} = 2\frac{kq^2}{a^2}\,(+\hat{\jmath})$$

$$\vec{F}_{NET} = \vec{F_1} + \vec{F_2} = \boxed{(-1.5\hat{\imath} + 2\hat{\jmath})\frac{kq^2}{a^2}}$$

$$\left|\vec{F}_{NET}\right| = \sqrt{1.5^2 + 2^2}\,\frac{kq^2}{a^2} = 2.5\frac{kq^2}{a^2}$$

$$\theta = \tan^{-1}\frac{2}{1.5} \approx 53°$$

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$$\begin{aligned} |F_1| &= k\frac{(2q)(3q)}{(2a)^2} \\ &= \frac{6}{4}\frac{kq^2}{a^2} \\ &= 1.5\,\frac{kq^2}{a^2} \end{aligned}$$

The object at the origin feels this force directed to the left, or in the $-\hat{x}$ direction. Second, the force on the object at the origin due to the object on the y-axis. The objects attract each other (since they have opposite charge) with a force equal to

$$\begin{aligned} |F_2| &= k\frac{(2q)(q)}{a^2} \\ &= 2\,\frac{kq^2}{a^2} \end{aligned}$$

The object at the origin feels this force directed in the positive $\hat{y}$ direction. To find the net force on the object at the origin, we add these two forces **as vectors**: $\vec{F}_{net} = \vec{F}_1 + \vec{F}_2$ In terms of the unit vectors:

$$\vec{F}_{net} = (-1.5\hat{i} + 2\hat{j})\frac{kq^2}{a^2} \tag{2}$$

We can also express our answer in terms of the magnitude and direction of the force. The magnitude of the net force is

$$\begin{aligned} |\vec{F}_{net}| &= \sqrt{1.5^2 + 2^2}\,\frac{kq^2}{a^2} \\ &= 2.5\,\frac{kq^2}{a^2} \end{aligned}$$

The direction of the force is at an angle θ above the negative x-axis equal to $tan(\theta) = 2/1.5$. So $\theta \approx 53°$ above the negative x-axis.

b) If the object at the origin were not present, what is the electric field at the origin due to the other two charges?

3

Since we know the force on and the charge of the object at the origin, we can determine the electric field. The magnitude of the electric field is $|\vec{F}|/|Q|$:

$$
\begin{aligned}
|\vec{E}| &= \frac{2.5\ kq^2/a^2}{2q} \\
&= 1.25\frac{kq}{a^2}
\end{aligned}
$$

Since the object at the origin has a positive charge, the direction of the electric field is in the same direction as the force on the object. So the direction of the electric field is approximately $53°$ above the negative x-axis.

Exercise 1.3
Two small "point" objects are separated by a distance d as shown in the figure. The object on the left has a net chare of $9q$, and the object on the right has a net charge of $-4q$, where q is some positive amount of charge. At what location in the figure is the electric field equal to zero?

The electric field will be zero when the individual electric field from the two objects cancel each other. The fields can cancel at points to the right of the $-4q$ charged object. This is because the object with $9q$ has more charge, and will need to be further away for the cancelation to occur. We let x be the distance from the location to the $-4q$ object. The electric field $\vec{E}_1$ at the location due to the $+9q$ object is

$$
\vec{E}_1 = \frac{k9q}{(x+d)^2}(+\hat{i}) \tag{3}
$$

The electric field $\vec{E}_2$ at the location due to the $-4q$ object is

$$
\vec{E}_2 = \frac{k4q}{x^2}(-\hat{i}) \tag{4}
$$

since the electric field at the location points towards the negative charge. The net electric field $\vec{E}_{net}$ is the sum of these two fields:

$$
\vec{E}_{net} = (+\frac{k9q}{(x+d)^2} - \frac{k4q}{x^2})\hat{i} \tag{5}
$$

Setting this equation equal to zero gives

$$\frac{k9q}{(x+d)^2} = \frac{k4q}{x^2} \tag{6}$$

We can solve for x by canceling the common factors and taking the square root of both sides:

$$\frac{3}{x+d} = \frac{2}{x}$$
$$3x = 2(x+d)$$
$$x = 2d$$

So at a distance of $2d$ to the right of the $-4q$ object, the attraction by the $-4q$ object equals the repulsion of the $+9q$ object. Note, that the result would be the same if q were an amount of negative charge.

Exercise 1.4

Extra electrons are placed on two light aluminum spheres of mass m that hang on thin threads as shown in the figure. The threads have a length of l and can be considered as massless. Gravity pulls the spheres down, but they repel each other since they have the same net charge. Let the opening angle at the top be θ.

a) If each sphere has the same charge q, find an expression for q in terms of m, g, k, l and θ.

The net force on the sphere is zero, since it is in equilibrium. The three forces are: gravity pointing downward, the electrostatic force to the side, and the tension towards the hanging point. The force of gravity has a magnitude of mg. The electrostatic force has a magnitude of kq^2/d^2, where d is the distance between the spheres. The three force vectors form a triangle, with the angle at the top being $\theta/2$. From the force triangle we have:

$$tan\,(\theta/2) = \frac{kq^2/d^2}{mg} \tag{7}$$

We use $\theta/2$ because the full opening angle is θ. The distance d is the distance between the two spheres. d can be expressed in terms of l and $\theta/2$: $sin(\theta/2) = (d/2)/l$, or $d = 2l\,sin(\theta/2)$. Substituting in for d in the first equation gives:

(1.3)
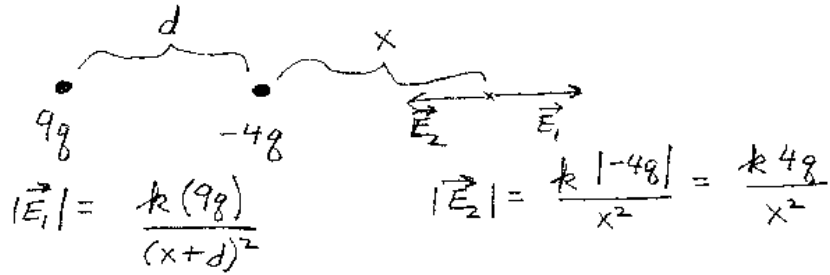
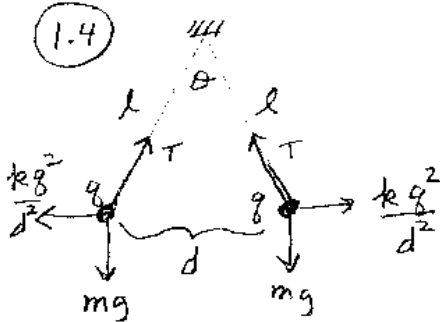$$|\vec{E_1}| = \frac{k(9q)}{(x+d)^2} \qquad |\vec{E_2}| = \frac{k|-4q|}{x^2} = \frac{k\,4q}{x^2}$$

$$\frac{k\,9q}{(x+d)^2} = \frac{k\,4q}{x^2}$$
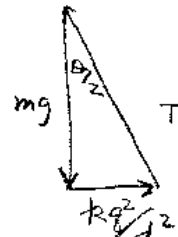
$$\frac{3}{x+d} = \frac{2}{x} \qquad \Rightarrow \qquad \boxed{x = 2d}$$

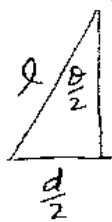$$3x = 2(x+d)$$

(1.4)

Forces:

$$\tan \theta/2 = \frac{k\,q^2}{mg\,d^2}$$

Geometry

$$\sin \theta/2 = \frac{d}{2l}$$

$$d = 2l \sin \theta/2$$

$$\tan \theta/2 = \frac{k\,q^2}{mg\,(2l\sin\theta/2)^2}$$

$$\boxed{q = 2l \sin \theta/2 \sqrt{\frac{mg}{k}\tan\theta/2}}$$

$$tan\ (\theta/2) = \frac{kq^2/(2l\ sin(\theta/2)^2}{mg}$$

$$tan\ (\theta/2) = \frac{kq^2}{mg4l^2\ sin^2\theta/2}$$

Solving for q we have

$$q = 2lsin(\theta/2)\sqrt{\frac{mgtan(\theta/2)}{k}} \qquad (8)$$

b) If the mass of the spheres is 0.5 grams, the length of the thread 10 cm, and the angle $\theta = 10°$, how many extra electrons are on each of the aluminum spheres?

First we can solve for q by plugging into the equation above.

$$q = 2(0.1)sin(5°)\sqrt{\frac{(0.0005)(9.8)tan(5°)}{9 \times 10^9}}$$

$$\approx 3.80 \times 10^{-9}\ Coulombs$$

Since each electron has a charge of 1.6×10^{-19} Coulombs, the number of electrons n equals

$$n = \frac{q}{e} = \frac{3.8 \times 10^{-9}}{1.6 \times 10^{-19}} \approx 2.4 \times 10^{10} \qquad (9)$$

elections.

Exercise 1.5
At a certain point in space an electron, which has a mass of 9.11×10^{-31} Kg, experiences a force of 3.2×10^{-19} Newtons in the $+x$ $(\hat{i})$ direction. What is the electric field at this point?

The magnitude of the electric field at a point in space equals the magnitude of the force divided by the magnitude of the charge of a particle if it is located at that point. The electrostatic force does not depend on the particles mass. Since the magnitude of the charge that an electron possesses is 1.6×10^{-19} Coulombs,

$$|\vec{E}| = \frac{|\vec{F_e}|}{|q|} = \frac{3.2}{1.6} = 2 \ N/C \tag{10}$$

The charge of an electron is negative. Since the force is in the $+x$ direction, the direction of the electric field is in the $-x$ direction:

$$\vec{E} = -2 \ \hat{i} \ N/c \tag{11}$$

Exercise 1.6
Suppose there are two small "point" objects located on the "y-axis". One object has a charge of $+q$ and is located at the coordinates (0,+a). The other "point" object has a charge of $-q$ and is located at the coordinates (0,-a). Find:

i: The electric field vector for locations on the y-axis (0,y) such that $|y| > a$.
ii: The electrid field vector for locations on the x-axis (x,0).

i: Calculating the electric field vector for locations on the y-axis is relatively easy since the object are "in line" with the y-axis. To find the net electric field we use Coulomb's law to find the electric field at the location from each of the objects. Then add the two electric field vectors (superposition principle) to find the net electric field. The electric field at the location (0,y) due to the positively charged object is

$$\vec{E_1} = \frac{kq}{(y-a)^2} \ (+\hat{j}) \tag{12}$$

The electric field at the location (0,y) due to the negatively charged object is
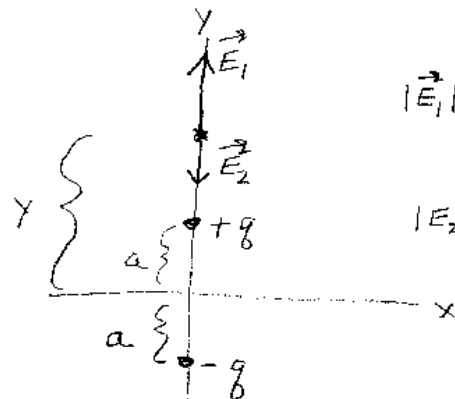
$$\vec{E_2} = \frac{kq}{(y+a)^2} \ (-\hat{j}) \tag{13}$$

The net electric field at the location (0,y) is just the vector sum of these two electric field vectors:

$$
\begin{aligned}
\vec{E}_{net} &= \vec{E_1} + \vec{E_2} \\
&= (\frac{kq}{(y-a)^2} - \frac{kq}{(y+a)^2})\hat{j}
\end{aligned}
$$

ii: Calculating the electric field vector for locations on the x-axis is more complicated than on the y-axis. This is because the electric field vectors from each object point in different directions.
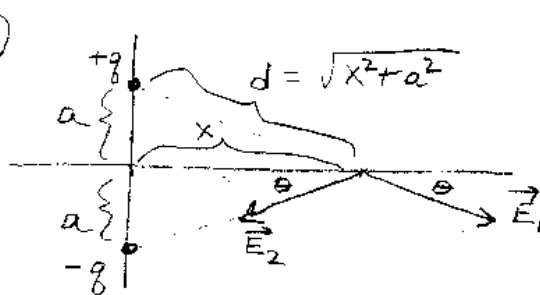
1.6

i)

$$|\vec{E_1}| = \frac{kq}{(y-a)^2}$$

$$|E_2| = \frac{kq}{(y+a)^2}$$

$$\vec{E}_{NET} = \frac{kq}{(y-a)^2}(+\hat{\jmath}) + \frac{kq}{(y+a)^2}(-\hat{\jmath})$$

$$= \boxed{\left(\frac{kq}{(y-a)^2} - \frac{kq}{(y+a)^2}\right)\hat{\jmath}}$$

ii)

$$d = \sqrt{x^2 + a^2}$$
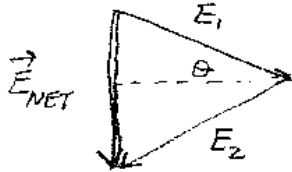
$$|E_1| = |E_2| = \frac{kq}{d^2}$$

$$= \frac{kq}{x^2 + a^2}$$

$$|E_{NET}| = 2|E_1|\sin\theta$$

$$= 2\frac{kq}{(x^2+a^2)}\frac{a}{d}$$

$$= \frac{2kqa}{(x^2+a^2)^{3/2}}$$

$$\vec{E}_{NET} = \boxed{\frac{2kqa}{(x^2+a^2)^{3/2}}(-\hat{\jmath})}$$

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The magnitude of the electric field at the location (x,0) due to the positively charged object is

$$|\vec{E}_1| = \frac{kq}{d^2} \tag{14}$$

where d is the distance from the location (x,0) to the position of the positive object (0,+a). Using Pythagorus' theorem we have $d^2 = x^2 + a^2$, so

$$|\vec{E}_1| = \frac{kq}{x^2 + a^2} \tag{15}$$

Similarly, the magnitude of the electric field at (x,0) due to the negatively charged object is:

$$|\vec{E}_2| = \frac{kq}{x^2 + a^2} \tag{16}$$

since d^2 also equals $d^2 = x^2 + a^2$. Although the magnitudes of $\vec{E}_1$ and $\vec{E}_2$ are equal, the direction of $\vec{E}_1$ is different than the direction of $\vec{E}_2$. $\vec{E}_1$ points away from the positively charged object and $\vec{E}_2$ points towards the negatively charged object. When we add the two vectors, $\vec{E}_1 + \vec{E}_2$, the x-components will cancel and the y-components will add.

$$\begin{aligned} \vec{E}_{net} &= \vec{E}_1 + \vec{E}_2 \\ &= 2\frac{kq}{x^2 + a^2} \, sin\theta(-\hat{j}) \end{aligned}$$

where θ is shown in the figure. From the figure we see that $sin\theta = a/d = a/\sqrt{x^2 + a^2}$. Substituting in for $sin\theta$ we have
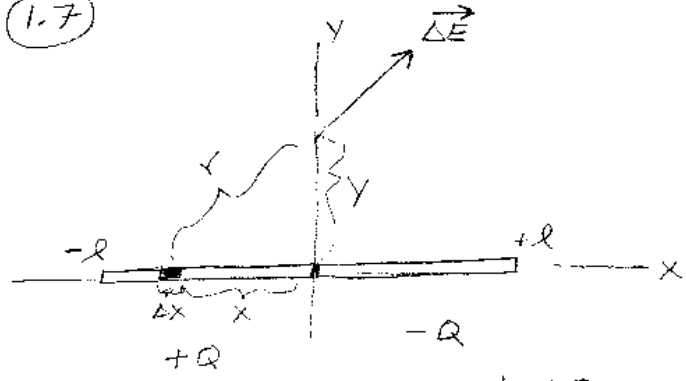
$$\vec{E}_{net} = \frac{2kqa}{(x^2 + a^2)^{3/2}}(-\hat{j}) \tag{17}$$

This setup of charges is called a "dipole".

Exercise 1.7
Consider the the charge distribution shown in the figure which consists of two uniformly charged thin rods, each of length l, attached end to end. The one on the left has a total charge of $-Q$ and the one on the right has a charge of $+Q$. They lie on the x-axis with their common end at $x = 0$. Find an expession for the electric field
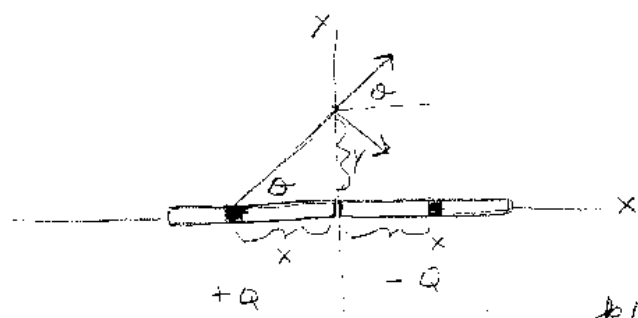
10

(1.7)

$$\frac{\Delta Q}{Q} = \frac{\Delta x}{\ell}$$

$$\Delta Q = \frac{Q}{\ell}\,\Delta x$$

$$|\Delta\vec{E}| = \frac{k\,\Delta Q}{r^2} = \frac{k\,\Delta Q}{x^2+y^2}$$

$$|\Delta\vec{E}| = \frac{kQ}{\ell}\,\frac{\Delta x}{(x^2+y^2)}$$

$$\Delta E_x = |\Delta\vec{E}|\cos\theta = \frac{kQ}{\ell}\,\frac{\Delta x}{(x^2+y^2)}\,\frac{|x|}{\sqrt{x^2+y^2}}$$

$$E_x = 2\int_{-\ell}^{0} \frac{kQ(-x)}{\ell}\,\frac{dx}{(x^2+y^2)^{3/2}} = \frac{2kQ}{\ell}\left((x^2+y^2)^{-\frac{1}{2}}\right)\Big|_{-\ell}^{0}$$

$$\boxed{E_x = \frac{2kQ}{\ell}\left(\frac{1}{y} - \frac{1}{\sqrt{\ell^2+y^2}}\right)}$$

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at locations (0,y) on the y-axis.

We cannot use Coulomb's "formula" with the rods as a whole, since they are not small "point" objects. We will have to divide up the rod into small little pieces and consider them as a collection of small point objects. Then we can apply Coulomb's Law for point objects and the principle of superposition. Lets divide up each rod into small pieces, each of length Δx. We will first find the electric field at the location (0,y) due to a little piece on the positive rod located at the coordinates (x,0), i.e. a distance x from the origin. The magnitude of the electric field $|\Delta \vec{E}|$ at (0,y) due to a point object at (x,0) is

$$|\Delta \vec{E}| = \frac{k\Delta Q}{x^2 + y^2} \tag{18}$$

where ΔQ is the charge in the small piece of the rod that has a length Δx. The vector $(\Delta \vec{E})$ will point away from pieces on the positive rod and towards pieces on the negative rod. Now we need to add up the electric field contributions at the location (0,y) from all the pieces on the x-axis. Since the rods have the same length, for every positive piece on the left rod there is a negative piece on the right rod that is the same distance from $x = 0$. When these two electric field vectors add, the vertical components will cancel and only the "x" components will add. Thus, we need only consider the x-component of $\Delta \vec{E}$ which we call ΔE_x. The x-component is:

$$
\begin{aligned}
\Delta E_x &= \frac{k\Delta Q}{x^2 + y^2} \, cos\theta \\
&= \frac{k\Delta Q}{x^2 + y^2} \frac{x}{\sqrt{x^2 + y^2}} \\
&= \frac{k\Delta Q \, x}{(x^2 + y^2)^{3/2}}
\end{aligned}
$$

In order to add up all the pieces (i.e. integrate), we need to express ΔQ in terms of Δx. Since the total charge on each rod is Q, we have

$$\frac{\Delta Q}{Q} = \frac{\Delta x}{l} \tag{19}$$

or $\Delta Q = (Q/l)\Delta x$. Substituting into the equation above we have

$$\Delta E_x = \frac{kQx\Delta x}{l(x^2 + y^2)^{3/2}} \tag{20}$$

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The integral over both rods equals twice the integral over one rod, so

$$
\begin{aligned}
E_x &= 2\int_{-l}^{0} \frac{kQx}{l(x^2+y^2)^{3/2}}\, dx \\
&= \frac{2Qk}{l}\left(\frac{1}{y} - \frac{1}{\sqrt{l^2+y^2}}\right)
\end{aligned}
$$

Exercise 1.8
Consider a cubical surface with sides of length 1 meter. The electric field on the left side is uniform with a magnitude of 5 N/C and is perpendicular to the side of the cube. The electric field on the right side is uniform with a magnitude of 10 N/C and is also perpendicular to the side of the cube. The electric field on the other surfaces are parallel to the surfaces. What is the net amount of charge that is enclosed by the cube?

We can use Gauss's Law to solve this problem because the surface is a closed surface. First we need to find the electric flux through each side of the cube. There is only electrix flux through the left and right sides. This is because for the other 4 sides the electric field on the surface is parallel to the surface. The electric flux through these sides equals zero.
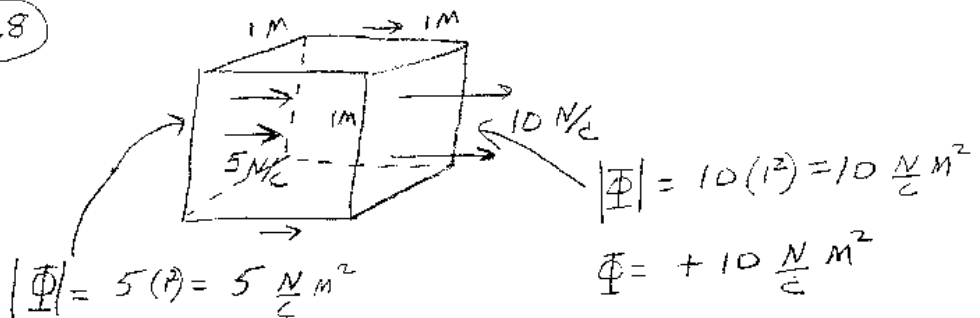
The magnitude of the electric flux through the left side equals $|\Phi_{leftside}| = (5\ N/C)(1\ M^2) = 5\ NM^2/C$. The sign of this flux is negative since it is into the closed surface: $\Phi_{leftside} = -5\ NM^2$.

Similarly, the magnitude of the electrix flux through the right side equals $\Phi_{rightside} = (10\ N/C)(1\ M^2) = +10NM^2/C$. The sign is positive since the electric field is directed out of the closed surface. Thus, the net electric flux equals $\Phi_{net} = 10-5 = 5\ NM^2/C$. From Gauss's law, the net electric flux is equal to the net charge enclosed by the surface divided by ϵ_0:

$$
\begin{aligned}
\Phi_{net} &= \frac{Q_{enclosed}}{\epsilon_0} \\
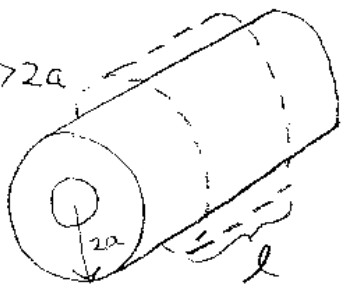5 &= \frac{Q_{enclosed}}{8.85\times 10^{-12}}
\end{aligned}
$$

Solving for $Q_{enclosed}$ gives $Q_{enclosed} \approx 4.4\times 10^{-11}$ Coulombs.
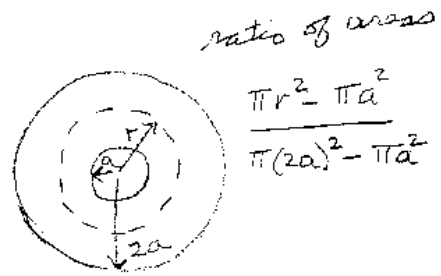
(1.8)

$|\Phi| = 10(1^2) = 10 \frac{N}{C} M^2$

$\Phi = +10 \frac{N}{C} M^2$

$|\Phi| = 5(1^2) = 5 \frac{N}{C} M^2$

$\Phi = -5 \frac{N}{C} M^2$

$\Phi_{NET} = 10 - 5 = 5 \frac{N}{C} M^2$

$\Phi_{NET} = \frac{Q_{inside}}{\epsilon_0} \Rightarrow Q_{inside} = 5\epsilon_0 \simeq 4.4 \times 10^{-11} C$

(1.9)

$r > 2a$

$0 + E(2\pi r \ell) + 0 = \frac{\lambda \ell}{\epsilon_0}$

$E = \frac{\lambda}{2\pi \epsilon_0 r}$

ratio of areas

$\dfrac{\pi r^2 - \pi a^2}{\pi(2a)^2 - \pi a^2}$

$a < r < 2a$

$0 + E(2\pi r \ell) + 0 = \frac{\lambda \ell}{\epsilon_0}\left(\dfrac{\pi r^2 - \pi a^2}{\pi(2a)^2 - \pi a^2}\right)$

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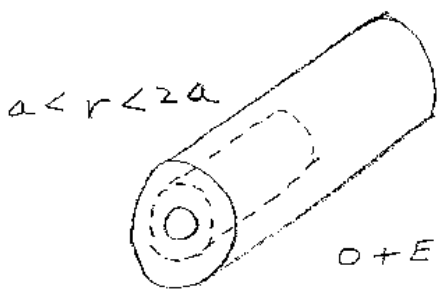
Exercise 1.9
You have been given an infinitely long uniformly charged thick pipe. The pipe is a non- conductor. That is, the charge does not move. The inner radius of the pipe is a, and the outer radius is $2a$. The amount of charge per length of the pipe is λ. Find the electric field vector for locations inside, within, and outside the uniformly charged pipe. (i.e. find $\vec{E}$ for $r < a$, $a < r < 2a$, and $r > 2a$ where r is the radial distance from the axis.

Because of the axial symmetry of the charge distribution, it is best to use Gauss's Law to determine the electric field at the different locations. The reason that Gauss's Law will be useful is that we can choose closed surfaces for which the net electric flux is easy to calculate. First let's figure out the shape of the surface we should use. We need to determine some properties of the electric field. Since there is axial symmetry and the pipe is infinite in length, the magnitude of the electric field at any position can only depend on the perpendicular distance from the position to the central axis. Since the pipe is infinite in length, the direction of the electric field must be radial from the central axis. Expressing this mathematically, we have $\vec{E}(r) = E(r)\hat{r}$, where r is the radial distance to the axis and $\hat{r}$ is a unit vector pointing away from the central axis.

Since the electric field has these symmetry properties, a most useful surface to choose is the surface of a "cylinderical can" whose axis is the same as the axis of the pipe. Let the radius of the can be r and its length be denoted by l. The electric flux through the top and bottom of the can will be zero since $\vec{E}$ is parallel to the top and bottom surfaces. The electric flux through the side of the can will equal to $E(r)$ times the surface area of the side of the can: $E(r)(2\pi rl)$.

$r \geq 2a$
If the radius r of the cylindrical surface is greater than $2a$ (the pipe), then all the charge of the pipe is within the surface. In this case, the radius of the can is larger than the pipe:

$$
\begin{aligned}
\Phi_{net} &= \frac{Q}{\epsilon_0} \\
0 + E(r)2\pi rl + 0 &= \frac{\lambda l}{\epsilon_0}
\end{aligned}
$$

The two zero's on the left side are because there is no flux through the top and bottom of the can. Solving for $E(r)$ we have

$$E(r) = \frac{\lambda}{2\pi r} \tag{21}$$

$a \leq r \leq 2a$

In this case the sides of the cylindrical surface are inside the pipe material. Only part of the charge is inside the surface. Expressing Gauss's Law we have

$$\Phi_{net} = \frac{Q_{inside}}{\epsilon_0}$$

$$0 + E(r)2\pi rl + 0 = \frac{Q_{inside}}{\epsilon_0}$$

where Q_{inside} is the charge inside the surface. Using ratios of the areas one has

$$Q_{inside} = Q\frac{\pi r^2 - \pi a^2}{\pi(2a)^2 - \pi a^2} \tag{22}$$

where Q is equal to λl. Substituting into the equation above yields

$$E(r) = \frac{\lambda(r^2 - a^2)}{6\pi ra^2} \tag{23}$$

$r \leq a$

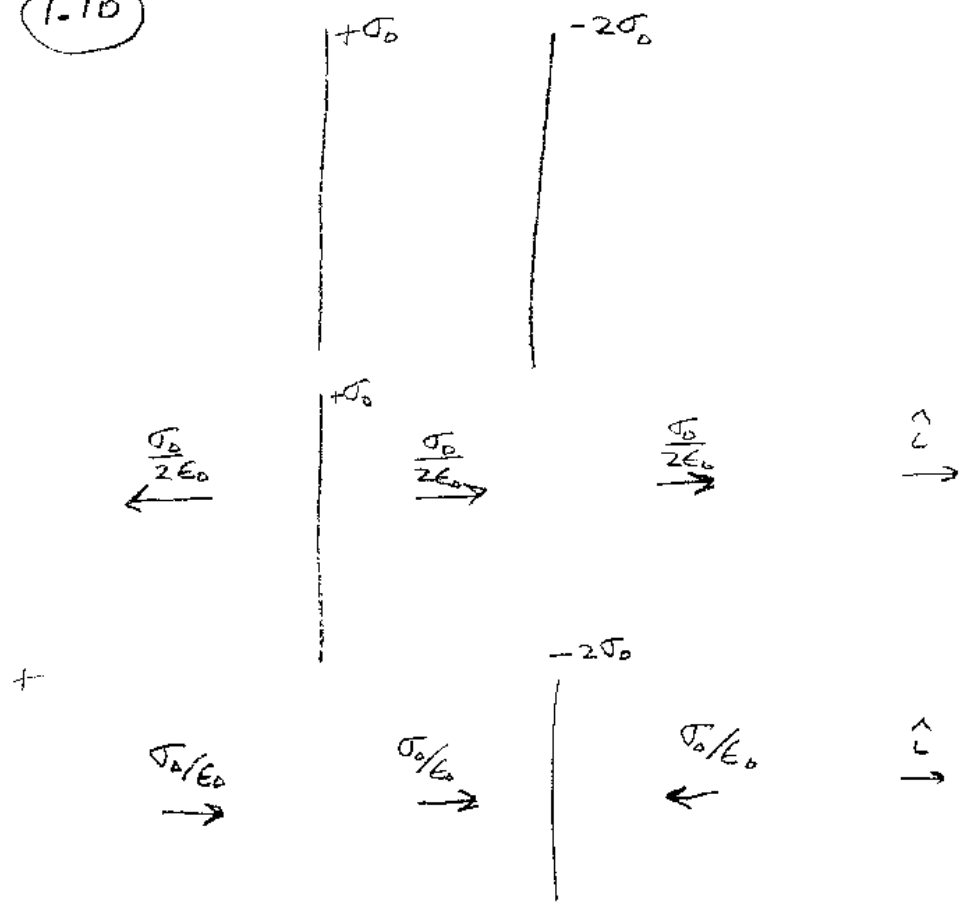In this case the cylinder is inside the "hole" in the pipe, so there is no charge within the cylinderical surface:

$$\Phi_{net} = \frac{Q_{inside}}{\epsilon_0}$$

$$0 + E(r)2\pi rl + 0 = \frac{0}{\epsilon_0}$$

$$E(r) = 0$$

The electric field is zero in the inner part of the pipe where there is no material.

Exercise 1.10

Consider two infinite planes of charge. Each of the planes are infinitely thin and are parallel to each other. The plans are separated by a distance a as shown in the figure. The plane on the left has a surface charge density of $+\sigma_0$. The plane on the right has a surface charge density of $-2\sigma_0$. Find an expression for the electric field in all

$\boxed{1.10}$

$+\sigma_0$ $-2\sigma_0$

$+\sigma_0$

$\dfrac{\sigma_0}{2\epsilon_0}$ $\dfrac{\sigma_0}{2\epsilon_0}$ $\dfrac{\sigma_0}{2\epsilon_0}$ $\hat{\imath}$

$\longleftarrow$ $\longrightarrow$ $\longrightarrow$ $\longrightarrow$

$-2\sigma_0$

$+$

σ_0/ϵ_0 σ_0/ϵ_0 σ_0/ϵ_0 $\hat{\imath}$

$\longrightarrow$ $\longrightarrow$ $\longleftarrow$ $\longrightarrow$

$-\dfrac{\sigma_0}{2\epsilon_0} + \dfrac{\sigma_0}{\epsilon_0}$	$\dfrac{\sigma_0}{2\epsilon_0} + \dfrac{\sigma_0}{\epsilon_0}$	$+\dfrac{\sigma_0}{2\epsilon_0} - \dfrac{\sigma_0}{\epsilon_0}$
$\parallel$	$\parallel$	
$\dfrac{\sigma_0}{2\epsilon_0}(+\hat{\imath})$	$\dfrac{3}{2}\dfrac{\sigma_0}{\epsilon_0}(+\hat{\imath})$	$\dfrac{\sigma_0}{2\epsilon_0}(-\hat{\imath})$
$\longrightarrow$	$\longrightarrow$	$\longleftarrow$

regions of space.

For this problem, we can use the result that we derived for the electric field from a single infinite plane of charge and the superposition principle. The plane on the left produces an electric field equal to $\sigma_0/(2\epsilon_0)$ which **points away** from the plane in all regions of space. The plane on the right produces an electric field equal to $2\sigma_0/(2\epsilon_0) = \sigma_0/\epsilon_0$ which **points towards** the plane in all regions of space. Superimposing these fields gives:

To the left of the two planes:

$$
\begin{aligned}
\vec{E} &= (-\frac{\sigma_0}{2\epsilon_0} + \frac{\sigma_0}{\epsilon_0})\hat{i} \\
&= \frac{\sigma_0}{2\epsilon_0}\hat{i}
\end{aligned}
$$

Between the two planes:

$$
\begin{aligned}
\vec{E} &= (+\frac{\sigma_0}{2\epsilon_0} + \frac{\sigma_0}{\epsilon_0})\hat{i} \\
&= \frac{3\sigma_0}{2\epsilon_0}\hat{i}
\end{aligned}
$$

To the right of the two planes:

$$
\begin{aligned}
\vec{E} &= (+\frac{\sigma_0}{2\epsilon_0} - \frac{\sigma_0}{\epsilon_0})\hat{i} \\
&= -\frac{\sigma_0}{2\epsilon_0}\hat{i}
\end{aligned}
$$

Note that these results are independent of a since the magnitude of electric field produced by an infinite plane of charge does not depend on the distance from the plane.

Exercise 2.1
What is the electric potential difference between the origin $(0,0)$ and the position $(2,3)$ meters if the electric field in this region is equal to $\vec{E}(\vec{r}) = E_0\hat{i}$, where E_0 has units of volts/meter?

To calculate the voltage difference between the locations P_1 amd P_2 we need to evaluate the integral $\int \vec{E} \cdot \vec{dr}$ from the position P_1 to P_2. Since the electric field is a conservative vector field, the result of this integral will be the same for any path we take. The easiest path to take is to first go in a straight line from $(0,0)$ to $(2,0)$, then in another straight line from $(2,0)$ to $(2,3)$.

The voltage difference along the path $(0,0) \rightarrow (2,0)$ is just $2E_0$ since the path is in the same direction as the electric field vector. The voltage difference along the path $(2,0) \rightarrow (2,3)$ will be zero, since the path is perpendicular to the direction of $\vec{E}$. So the voltage difference between the origin and $(2,3)$ is just $2E_0$ with the origin being at the higher potential than $(2,3)$.

Exercise 2.2
Two small objects, each having a charge of $-Q$, are located on the y-axis a distance $+a$ and $-a$ from the origin. Another small object, having a charge q, starts at rest on the x-axis very far away (i.e. at $x = \infty$). The two objects on the y-axis are fixed in place. The object of charge q is released from rest, is attracted to the other two objects, and travels along the x-axis. What is its speed as it passes through the origin?

This problem can be solved fairly easily using the work-energy theorum. The net electrostatic work done on "q" will equal the change in its kinetic energy. The net electrostatic work done on the object equals the potential difference times its charge: $W_{net} = q\Delta V$. The electric potential at $x = \infty$ is zero. The electric potential at the origin is $-kQ/a - kQ/a = -2kQ/a$. So, potential difference between $x = \infty$ and $x = 0$ is

$$\Delta V = 0 - (-\frac{2kQ}{a}) = \frac{2kQ}{a} \tag{24}$$

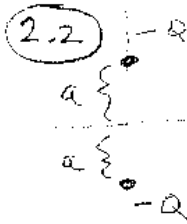The net work done on "q" is $W_{net} = 2kQq/a$. The work-energy theorem yields

$$\frac{mv^2}{2} - 0 = \frac{2kQq}{a}$$
$$v = 2\sqrt{\frac{kQq}{ma}}$$

as the speed of the object as it passes through the origin.

Exercise 2.3
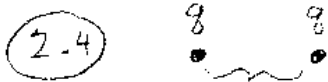Calculate the amount of energy that it takes to ionize a hydrogen atom. That is,
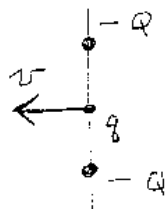
2.2

$-Q$

a {

$-q_0$
at
$r = \infty$
$v_0 = 0$

a {

$-Q$

Before $\quad E_{TOT} = 0 + 0 = 0$

$0 = E_{TOT} = \dfrac{m}{2}v^2 - \dfrac{kq_0 Q}{a} - \dfrac{kq_0 Q}{a}$

$-Q$

$v \leftarrow q$

$-Q$

$0 = \dfrac{m}{2}v^2 - 2\dfrac{kq_0 Q}{a}$

$$\boxed{v = 2\sqrt{\dfrac{kq_0 Q}{ma}}}$$

2.4

$q_0 \quad\quad q_0$

$\overset{r}{\frown}$

Before $\quad E_{TOT} = \dfrac{kq_0^2}{r} + 0$

$\uparrow$

NO K.E.

$\overset{v_f}{\longleftarrow} \quad \infty \quad \overset{v_f}{\longrightarrow}$

after $\quad E_{TOT} = 0 + 2\dfrac{mv_f^2}{2}$

$\uparrow$

$U_{electrostatic} = 0$

$2\ \dfrac{mv_f^2}{2} = \dfrac{kq_0^2}{r}$

$$\boxed{v_f = q_0 \sqrt{\dfrac{k}{mr}}}$$

what is the total amount of energy that the electron possesses when it orbits a proton? Express your answer in terms of e, k, and r where r is the radius of the orbit. Assume that the electron orbits in a circle about the proton.

If an electron and proton are separated by a distance r, the electrostatic potental energy is:

$$
\begin{aligned}
U_{electrostatic} &= \frac{kq_1q_2}{r} \\
&= -\frac{ke^2}{r}
\end{aligned}
$$

The kinetic energy is given by $K.E. = mv^2/2$. For circular motion, the magnitude of the net force on the electron is $F = mv^2/r$. This force is supplied by the electrostatic interaction:

$$
\begin{aligned}
\frac{mv^2}{r} &= k\frac{e^2}{r^2} \\
mv^2 &= k\frac{e^2}{r}
\end{aligned}
$$

Substituting in these expressions for the total energy we have:

$$
\begin{aligned}
E_{tot} &= K.E. + P.E. \\
&= \frac{mv^2}{2} - \frac{ke^2}{r} \\
&= \frac{ke^2}{2r} - \frac{ke^2}{r} \\
&= -\frac{ke^2}{2r}
\end{aligned}
$$

This is an important expression. By measuring the energy of ionization, E_{tot}, we can determine r, which is the radius of the hydrogen atom. The ionization energy of hydrogen is 2.18×10^{-18} Joules. Using $k = 9 \times 10^9$ and $e = 1.6 \times 10^{-19}$ gives $r = 0.529 \times 10^{-10}$ meters as the radius of the hydrogen atom.

Exercise 2.4
Two small objects, each having a net charge of q and mass m, are initially held a

distance r apart. Then they are released and repel each other apart. What is the final speed they have when they are infinitely far apart?

For this problem we can use the work energy theorem. The net work done on the system will be the change in the system's kinetic energy. The net work done is the change in the systems electrostatic energy which is the initial potential energy minus the final potential energy:

$$W_{net} = k\frac{q^2}{r} - 0 = k\frac{q^2}{r} \tag{25}$$

The system's kinetic energy is the sum of the kinetic energy of the two objects: $K.E. = mv_f^2/2 + mv_f^2/2 = mv_f^2$, where v_f is the final speed of an object (when it is infinitely far away from the other object). Thus, we have

$$
\begin{aligned}
\frac{mv_f^2}{2} + \frac{mv_f^2}{2} &= k\frac{q^2}{r} \\
mv_f^2 &= k\frac{q^2}{r} \\
v_f &= q\sqrt{\frac{k}{mr}}
\end{aligned}
$$

Exercise 2.5
Consider a uniformly charged sphere that has a total charge of Q that has a radius of R. What is the electrostatic potential difference, ΔV, between the center of the sphere and the surface of the sphere?

To find an expression for the potential difference between these two spots, we need to integrate the electric field from the center to the surface of the sphere: $\Delta V = \int \vec{E} \cdot \vec{dr}$. We first need to find an expression for the electric field inside the uniformly charged sphere. Because there is spherical symmetry, Gauss's Law will be useful for us.

Since the charge distribution has spherical symmetry, **the electric field vector must point radially away from the center of the sphere**, and **the magnitude of the electric field can only depend on r**, where r is the distance to the sphere's center. Thus, $\vec{E}(\vec{r}) = E(r)\hat{r}$, where $r = |\vec{r}|$. Because of this spherical symmetry, a useful surface to choose is a spherical shell. On the surface of the shell, the electric
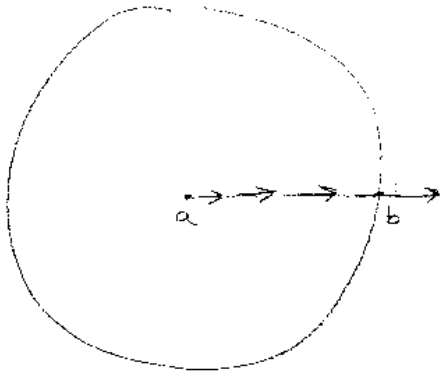
(2.5)

$$4\pi r^2 E = \frac{Q_{inside}}{\epsilon_0}$$
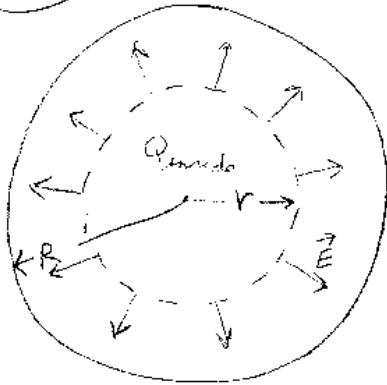
$$\frac{Q_{inside}}{Q} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} = \frac{r^3}{R^3}$$

$$4\pi r^2 E = \frac{Q r^3}{\epsilon_0 R^3}$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{Qr}{R^3} = \frac{kQr}{R^3}$$

$$\Delta V = \int_a^b \vec{E} \cdot \vec{dr}$$

$$= \int_0^R \frac{kQr}{R^3} \, dr$$

$$V_a - V_b = \frac{kQ}{R^3} \frac{r^2}{2} \Big|_0^R = \boxed{\frac{kQ}{2R}}$$

field is perpendicular to the surface and has the same magnitude at any point on the surface. The surface integral of $\vec{E} \cdot d\vec{A}$ is easy to calculate:

$$\int\int \vec{E} \cdot d\vec{A} = \frac{Q_{inside}}{\epsilon_0}$$
$$4\pi r^2 E = \frac{Q_{inside}}{\epsilon_0}$$

For points inside the sphere we choose our spherical shell to be inside the sphere, that is $r < R$. In this case, the amount of charge inside the shell is just the volume inside the shell times the charge density: $Q_{inside} = (4/3)\pi r^3 \rho$ where ρ is the charge density. Since the charge density of the sphere is $\rho = Q/((4/3)\pi R^3)$, we have

$$Q_{inside} = Q\frac{r^3}{R^3} \tag{26}$$

Substituting into the expression of Gauss's Law above yields

$$4\pi r^2 E = \frac{Q_{inside}}{\epsilon_0}$$
$$4\pi r^2 E = Q\frac{r^3}{R^3 \epsilon_0}$$
$$E = \frac{Qr}{4\pi\epsilon_0 R^3}$$
$$E = kQ\frac{r}{R^3}$$

which is the magnitude of the electric field for points inside the uniformly charged sphere .

Now we can find the potential difference between the center and a point on the surface.

$$\Delta V = \int \vec{E} \cdot d\vec{r}$$
$$= \int_0^R kQ\frac{r}{R^3}\, dr$$
$$= \frac{kQ}{2R}$$

Exercise 2.6
Four small "point" objects are located at the corners of a square of side L. Each object has a charge q. Find an expression for the electric potential for locations on the axis that goes through the center of the square and perpendicular to its surface.

Let the z-axis go through the center of the square and perpendicular to the surface. We need to find $V(z)$. The electric potential a distance r away from a "point" charge is kq/r. We need to find the distance r away from a point on the z-axis to a corner of the square. From the figure, we can see that $r = \sqrt{z^2 + L^2/2}$. Thus, the electric potential at locations on the z-axis due to one of the charges is $V(z) = kq/\sqrt{z^2 + L^2/2}$. Since V is a scalar, we can just add the electric potential from the four charges:

$$V(z) = \frac{4kq}{\sqrt{z^2 + L^2/2}} \tag{27}$$

From this expression we can calculate the electric field in the "z" direction by differentiating $V(z)$ with respect to z:

$$
\begin{aligned}
E_z &= -\frac{\partial V}{\partial z} \\
&= \frac{4kqz}{(z^2 + L^2/2)^{3/2}}
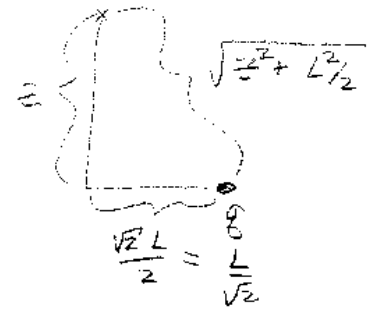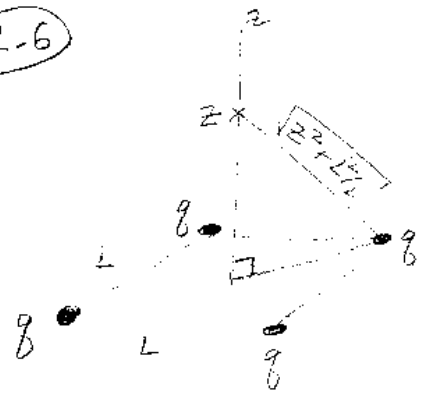\end{aligned}
$$

Note that this is the same expression we obtained in a homework problem by adding up the four electric field vectors. Taking the derivative of the potential is much easier.

Exercise 2.7
Consider a spherical conducting shell that has an inner radius a and an outer radius b. The inside of the conductor is hollow ($r < a$), and there is a small object of charge $-2Q$ at the center. The net charge on the conductor is $+Q$. Find an expression for the electric field everywhere, and the location of charge on the conducting shell.

Inside the conductor $a < r < b$ the electric field is zero, since this region is inside a conductor. To find the electric field at other locations, we can use Gauss's Law because there is spherical symmetry. Since the conductor and charge have spherical symmetry, the electric field must be radial and its magnitude can only depend on r, the distance to the center of the conductor. That is, $\vec{E}(\vec{r}) = E(r)\hat{r}$. Thus, if we choose a spherical surface to do our surface integral, we have:

25

(2-6)

$$V(z) = \frac{4kq}{\sqrt{z^2 + L^2/2}}$$

(2.7)
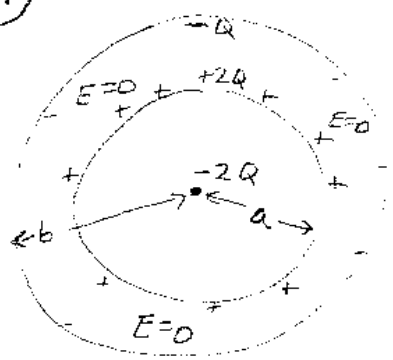
$a < r < b$ is inside the conductor.

$$\boxed{E = 0}$$

$r > b$

$$4\pi r^2 E = \frac{-2Q + Q}{\epsilon_0}$$

$$\vec{E} = \frac{kQ}{r^2}(-\hat{r})$$

$r < a$

$$4\pi r^2 E = \frac{(-2Q)}{\epsilon_0}$$

$$\vec{E} = \frac{2kQ}{r^2}(-\hat{r})$$

$$\oint \oint \vec{E} \cdot \vec{dr} = \frac{Q_{inside}}{\epsilon_0}$$

$$4\pi r^2 E = \frac{Q_{inside}}{\epsilon_0}$$

$$\vec{E} = \frac{kQ_{inside}}{r^2}\hat{r}$$

where Q_{inside} is the charge inside a sphere of radius r. This expression is valid for all locations when there is spherical symmetry in the charge distribution.

$r > b$

For $r < b$ the amount of charge less than r is $Q_{inside} = Q - 2Q = -Q$. So:

$$\vec{E} = \frac{kQ}{r^2}(-\hat{r}) \tag{28}$$

$r < a$

For $r < a$, the amount of charge less than r is $-2Q$. So:

$$\vec{E} = \frac{k2Q}{r^2}(-\hat{r}) \tag{29}$$

For the electric field to be zero inside the conductor, there must be a net charge of $+2Q$ on the inner surface at $r = a$. This leaves a total charge of $-Q$ on the outer surface of the conductor at $r = b$.

Exercise 2.8

Charges are arranged to produce an electric field given by:

$$\vec{E}(x, y, z) = cx\hat{i} + d\hat{j} \tag{30}$$

where c and d are constants. How much charge is contained inside a cube with sides of length a that has one corner at the origin and whose sides are parallel to the x-, y-, and z-planes?

To find the charge inside the cubical surface it is best to use Gauss's Law since we have a closed surface. Gauss's Law states that the net electric flux through a closed surface equals the net charge enclosed by the surface. We need to find the electric flux through each of the 6 sides.

Sides parallel to the x-y plane:
For these two sides the area vector points in the $\pm z$ direction. For the top side of the cube, $\vec{A} = a^2\hat{k}$, and for the bottom side, $\vec{A} = a^2(-\hat{k})$. Since the electric field does not have any component in the z-direction (i.e. in the $\hat{k}$ direction) $\vec{E} \cdot \vec{A}$ is zero for these two sides.

Sides parallel to the x-z plane:
For these two sides the area vector points in the $\pm y$ direction. For the back side of the cube, $\vec{A} = a^2\hat{j}$, and for the front side of the cube, $\vec{A} = a^2(-\hat{j})$. The y-component of $\vec{E}$ is constant on the surface, so the flux through the back side is $\Phi = da^2$. The flux through the front side is $\Phi = -da^2$, where the minus sign is because $\vec{A}$ and $\vec{E}$ are in opposite directions. The y-component of the electric field is pointing into the cube. Adding these together gives $\Phi = da^2 - da^2 = 0$'

Sides parallel to the y-z plane:
For these two sides, the area vector points in the $\pm x$ direction. For the left side of the cube, $x = 0$ and therefore $E_x = c(0) = 0$. Since $E_x = 0$ on this side the electric flux through the surface is also zero. On the right side however, $E_x = ca$ since here $x = a$. Thus, the flux through the right side of the cube equals $\Phi = ca(a^2) = ca^3$. Adding up the flux from all 6 sides gives:

$$\Phi_{net} = \frac{Q_{inside}}{\epsilon_0}$$

$$0 + 0 + da^2 - da^2 + 0 + ca^3 = \frac{Q_{inside}}{\epsilon_0}$$

$$Q_{inside} = ca^3\epsilon_0$$

which is the total amount of charge contained inside of the cubical surface.

(2.8)

$|\vec{A}| = a^2$

$\vec{A}$ is $\perp$ to the surface.
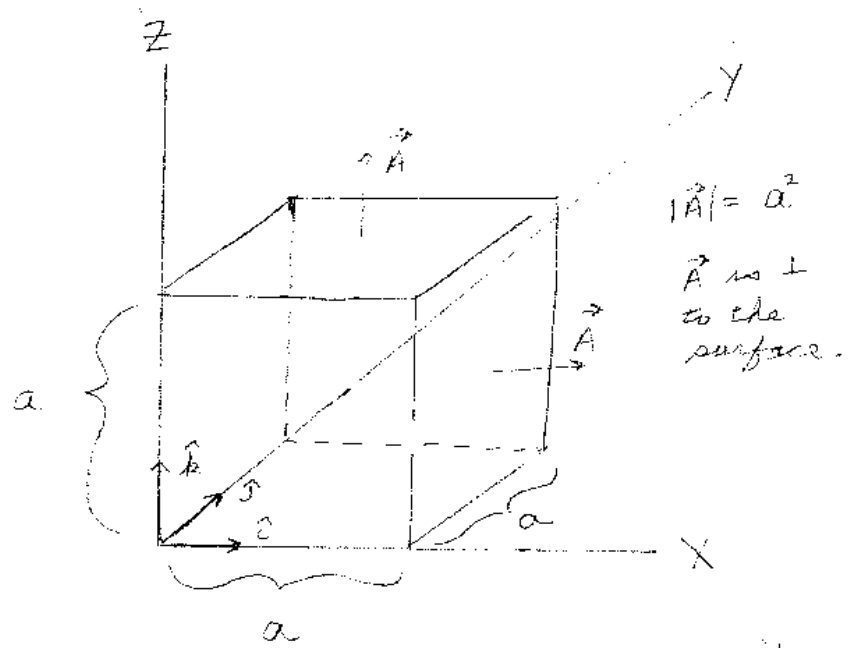
There are 6 area vectors, one on each side of the cubical surface.