

Exercises on Fluids and Gravity

Exercise 1.1

The Empire State Building in New York is around 320 meters tall. If we assume that the density of air is 1.2 Kg/m^3 throughout the height of the building, what is the difference in air pressure from the top to the bottom of the building?

The difference in pressure ΔP from the top to the bottom is just $\Delta P = \rho gh$, where ρ is the density of air. Plugging into this formula gives:

$$\begin{aligned}\Delta P &= \rho_{air}gh \\ &= \left(1.2 \frac{\text{kg}}{\text{m}^3}\right)9.8 \frac{\text{m}}{\text{s}^2}320\text{m} \\ &\approx 3760 \frac{\text{N}}{\text{m}^2}\end{aligned}$$

since atmospheric pressure is around $1 \times 10^5 \text{ N/m}^2$, this pressure difference is around 3.8% of one atmosphere. This is a small difference, and we don't notice it.

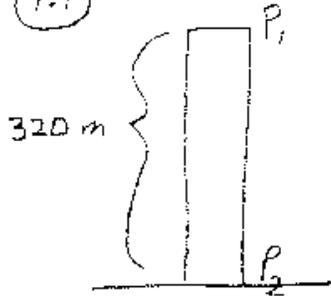
Exercise 1.2

Heidi has a swimming pool that she fills with water and oil. The water is at the bottom of the pool and has a depth of 2 meters. The oil is 1 meter thick, and since its density is 920 kg/m^3 floats on top of the water. What is the pressure difference between the surface of the oil and the bottom of the pool? That is, what is the pressure difference between the points A and B as shown in the figure?

The change in pressure $\Delta P = \rho gh$ is only valid if the density of the fluid doesn't change. Let C be a point at the interface between the water and the oil. In this problem, it is best to find the pressure difference between A and C , then add the pressure difference between C and B :

$$\begin{aligned}\Delta P &= \rho_{oil}gh_{oil} + \rho_{water}gh_{water} \\ &= 920(9.8)(1) + 1000(9.8)(2) \\ &= 28616 \frac{\text{N}}{\text{m}^2}\end{aligned}$$

1.1

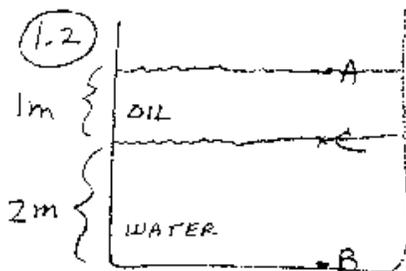


$$P_2 - P_1 = \rho g h$$

$$= \left(1.2 \frac{\text{kg}}{\text{m}^3}\right) (9.8 \frac{\text{m}}{\text{s}^2}) (320 \text{m})$$

$$\approx \boxed{3760 \text{ N/m}^2}$$

1.2



$$P_C - P_A = \rho_{\text{oil}} g h$$

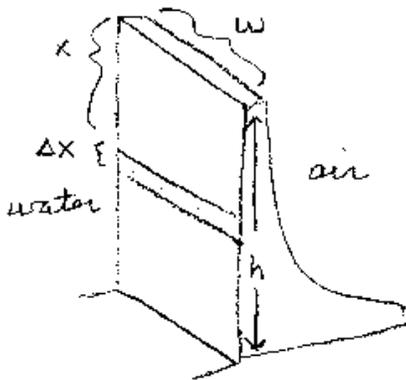
$$= 920 (9.8) (1) = 9016 \frac{\text{N}}{\text{m}^2}$$

$$P_B - P_C = \rho_{\text{water}} g h$$

$$= 1000 (9.8) (2) = 19600 \frac{\text{N}}{\text{m}^2}$$

$$P_B - P_A = 9016 + 19600 = \boxed{28616 \text{ N/m}^2}$$

1.3



$$\Delta F = (\text{Pressure}) (\Delta A)$$

$$\Delta F = (\rho g x) (w \Delta x)$$

$$\Delta F = \rho g w x \Delta x$$

$$F = \int_0^h \rho g w x dx$$

$$F = \boxed{\frac{\rho g w h^2}{2}}$$

Exercise 1.3

What is the total force on the dam surface shown in Fig 1.3. The dam has a height h , and a width w .

We cannot just multiply the pressure, ρgh , times the area lw to find the total force on the dam surface. This is because the pressure is not constant on the dam surface. The pressure increases with depth. To calculate the total force, we need to divide up the dam area into horizontal strips. Let x be the depth of a strip, and Δx be the small height of the strip. The strip will have a width of length w as shown in the figure.

Note that the pressure on each strip is essentially constant, since the strips lie horizontally. At a depth x from the surface, the pressure is ρgx . Therefore, the force ΔF on a strip located at a depth x from the surface is $\Delta F = Pw\Delta x = (\rho gx)w\Delta x$. To find the total force, we just add up (i.e. integrate) the forces on the strips:

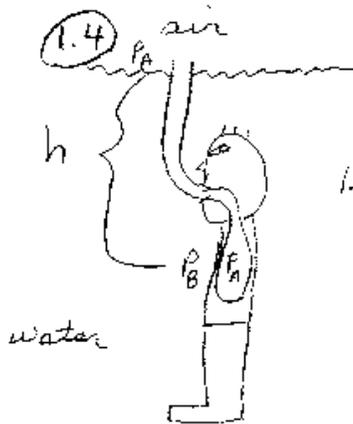
$$\begin{aligned} F &= \int_0^h \rho gxw \, dx \\ &= \rho gw \frac{x^2}{2} \Big|_0^h \\ &= \frac{\rho gwh^2}{2} \end{aligned}$$

This expression can also be written as $(\rho gh/2)wh$, which is just the average pressure times the area. Also note that the net force is equal to the difference in the force due to the water on one side and the air on the other side of the dam. The air pressure on the side of the dam is essentially the same as the pressure at the surface of the water. That is why the difference in pressure is just $\Delta P = \rho gx$.

Exercise 1.4

Human lungs are able to push against a pressure difference of around one tenth of an atmosphere. Suppose you wanted to breath underwater using a hose that reached to the surface. How far below the surface can you go?

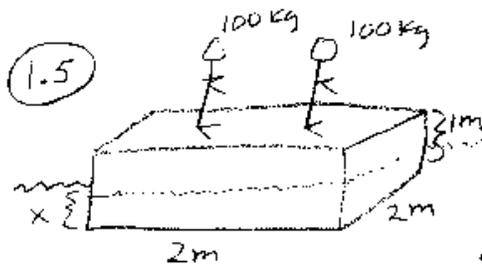
As we see in the figure, if the hose reaches up to the surface, then the air pressure inside your lungs is one atmosphere. However, on the outside, the pressure is one atmosphere plus the increased pressure due to the water. Let h be the depth that your lungs are under the surface of the water. Then the pressure just outside your lungs is one atmosphere plus $\rho_{water}gh$. Therefore, the difference in pressure is $\rho_{water}gh$.



$$P_B - P_A = \rho g h$$

$$1.013 \times 10^4 = (1000 \frac{\text{kg}}{\text{m}^3}) (9.8 \frac{\text{m}}{\text{s}^2}) h$$

$$h \approx 1 \text{ meter}$$

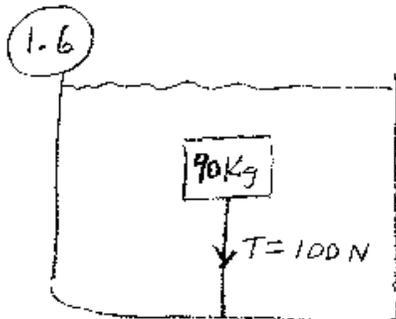


TOTAL = BOUYANT FORCE
WEIGHT

$$(2)(2)(1)(700) + 200 = 1000(4)x$$

mass of raft mass of passengers mass of DISPLACED WATER

$$x = \frac{3000}{4000} = \boxed{.75 \text{ m}}$$



$$\text{Buoyant Force} = 90(9.8) + 100 = 982 \text{ N}$$

$$\rho_{\text{H}_2\text{O}} g V = 982$$

$$V = \frac{982}{1000(9.8)} = 0.1 \text{ m}^3$$

$$\rho = \frac{m}{V} = \frac{90}{0.1} = \boxed{900 \frac{\text{kg}}{\text{m}^3}}$$

This cannot be greater than 1/10 atmosphere, which is 1.013×10^4 Pa. Now we can solve for the maximum value that h can be:

$$\begin{aligned}\Delta P &= \rho_{water}gh \\ 1.013 \times 10^4 \text{ Pa} &= 1000 \frac{\text{kg}}{\text{m}^3} 9.8 \frac{\text{m}}{\text{s}^2} h\end{aligned}$$

Solving for h gives $h \approx 1 \text{ m}$. That is, you can only dive to a depth of around 3 feet using this method of breathing. To dive deeper, you need to breath in air at a higher pressure. A scuba tank of compressed air plus a regulator adjust the pressure of the air you breathe in to be equal to the pressure of the water just outside your lungs.

Exercise 1.5 You have built a big raft out of light wood, which has a density of 700 kg/m^3 . Your raft has dimensions of 2 meters by 2 meters, and is 1 meter thick. You put the raft into your swimming pool and both you and your friend sit on it. Both you and your friend each have a mass of 100 Kg. How far into the water does the raft "sink" when you and your friend are on it? That is, what is x in the figure?

Since the raft is floating, the bouyant force on the raft must equal to total weight of the raft plus its two passengers. Let x be the depth that the raft sinks into the water. The bouyant force is the weight of the water displaced:

$$\begin{aligned}F_{bouyant} &= V_{displaced} \rho_{water} g \\ &= (2)(2)x(1000)g \\ &= 4000xg\end{aligned}$$

The total weight of the raft plus passengers is:

$$\begin{aligned}W_{total} &= (2)(2)(1)(700)g + 200g \\ &= 3000g\end{aligned}$$

Since the total weight must equal the bouyant force, we can solve for x :

$$\begin{aligned}4000xg &= 3000g \\ x &= \frac{3000}{4000} \\ x &= 0.75 \text{ meters}\end{aligned}$$

So the raft sinks into the water 75 centimeters. Note that the raft by itself (without any passengers) will "sink" into the water only 70 centimeters.

Exercise 1.6

A 90 Kg object will float in water. It is tied down to the bottom of a pool of water as shown in the figure. The tension in the rope that is holding it under water is 100 Newtons. What is the density of the object?

The principle of physics that is important here is Archimede's Principle: The bouyant force is equal to the weight of the fluid displaced. In this example, the bouyant force $F_{bouyant}$ must equal the objects weight plus 100 Newtons:

$$\begin{aligned}F_{bouyant} &= mg + 100 N \\ &= 90(9.8) + 100 \\ &= 982 \text{ Newtons}\end{aligned}$$

However, the bouyant force is equal to the weight of the fluid displaced. Let V be the volume of the object. Then the bouyant force equals $\rho_{fluid}gV$. Setting this expression equal to 982 Newtons gives:

$$\begin{aligned}\rho_{water}gV &= 982 \\ 1000(9.8)V &= 982 \\ V &\approx 0.1 m^3\end{aligned}$$

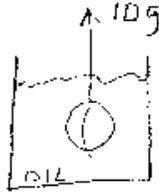
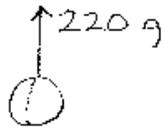
So the density of the object is

$$\rho_{object} = \frac{90 \text{ kg}}{0.1 m^3} = 900 \frac{\text{kg}}{m^3} \tag{1}$$

Exercise 1.7

Karen wants to determine the density of a light rock she found. She suspends the rock with a string from a balance. In the air, the balance reads 220 grams. Then she completely submerges the rock in a glass of oil while it is still suspended from the balance. The balance now reads 10 grams. She knows that the density of the oil is $850 \text{ kg}/m^3 = 0.85 \text{ g}/cm^3$. What is the density of the rock?

1.7



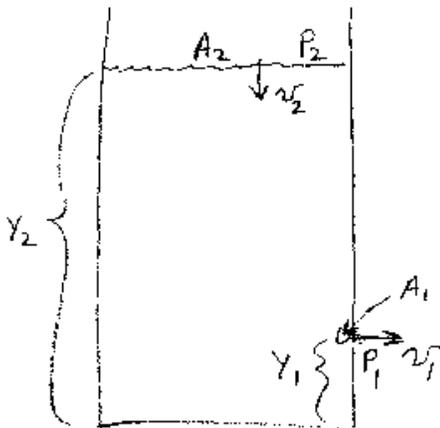
$F_{BOYANT} = \text{wt of fluid displaced}$

$$(220 - 10)g = \rho_{\text{dil}} g V$$

$$V = \frac{210}{0.85} \approx 247 \text{ cm}^3$$

$$\rho_{\text{rock}} = \frac{220 \text{ gms}}{247 \text{ cm}^3} \approx \boxed{0.89 \text{ g/cm}^3}$$

1.8



$$v_2 A_2 = v_1 A_1$$

$$v_2 = v_1 \left(\frac{A_1}{A_2} \right)$$

$$\frac{\rho}{2} v_2^2 + \rho g y_2 + P_2 + P_2 =$$

$$\frac{\rho}{2} v_1^2 + \rho g y_1 + P_1$$

$$\frac{\rho}{2} v_2^2 + \rho g y_2 \approx \frac{\rho}{2} v_1^2 + \rho g y_1$$

$$v_1 = \sqrt{\frac{2g(y_2 - y_1)}{1 - (A_1/A_2)^2}}$$

The principle of physics that is important is Archimede's Principle. The bouyant force equals the weight of the fluid displaced. From our balance readings, we know that the bouyant force is $(220 - 10)g$ where g is the acceleration due to gravity. The weight of the fluid displaced is equal to $\rho_{fluid}gV$, where V is the volume of the rock. Note that we use the whole volume of the rock because the rock is **completely submerged**. Equating these two quantities yields:

$$\begin{aligned}\rho_{fluid}gV &= (220 - 10)g \\ 0.85gV &= 210g \\ V &= 210/0.85 \\ V &\approx 247 \text{ cm}^3\end{aligned}$$

So the density of the rock is $\rho_{rock} \approx 220/247 \approx 0.89 \text{ g/cm}^3$. This is the method that geologists can use to measure the density of irregularly shaped rocks.

Exercise 1.8

Consider the tank of water shown in the figure. The tank has a small hole in the side located a distance y_1 from the bottom. The area of the hole is A_1 . The cross sectional area of the tank is A_2 , and the water level is a height y_2 from the bottom. Find an expression for the speed of the water, v_1 as it exits the hole in the tank. Assume that the water is an incompressible fluid.

Let v_2 be the speed of the water level at the top as the water runs out of the tank. There are two principle's of physics we can consider. Since the water is incompressible, we must have:

$$v_1A_1 = v_2A_2 \tag{2}$$

from the continuity equation.

The second is Bernouli's principle which states that $(\rho/2)v^2 + \rho gy + P$ is a constant throughout the fluid. Let point "1" be at the opening of the hole in the tank, and let point "2" be a point at the top of the water. Then, Bernouli's principle yields:

$$\frac{\rho}{2}v_2^2 + \rho gy_2 + P_2 = \frac{\rho}{2}v_1^2 + \rho gy_1 + P_1 \tag{3}$$

However, the pressure at point 1 is essentially the same as the pressure at point 2. This is because both points are "touching" the atmosphere, and are at atmospheric

pressure. The difference in height between point "1" and point "2" times the density of air times g is very small. So, $P_1 \approx P_2$, and the above equation simplifies to

$$\frac{\rho}{2}v_2^2 + \rho g y_2 = \frac{\rho}{2}v_1^2 + \rho g y_1 \quad (4)$$

From the continuity equation we have $v_2 = v_1(A_1/A_2)$. Substituting this expression for v_2 into Bernoulli's equation gives:

$$\frac{\rho}{2}\left(v_1 \frac{A_1}{A_2}\right)^2 + \rho g y_2 = \frac{\rho}{2}v_1^2 + \rho g y_1 \quad (5)$$

Dividing by ρ and solving for v_1 yields

$$v_1 = \sqrt{\frac{2g(y_2 - y_1)}{1 - (A_1/A_2)^2}} \quad (6)$$

Note that if the area of the hole is much smaller than the cross sectional area of the tank, i.e. $(A_1/A_2) \ll 1$, that $v_1 \approx \sqrt{2g(y_2 - y_1)}$. This is the speed that an object would obtain if dropped from rest at y_2 and fell to a height y_1 .

Exercise 1.9

Air exerts a bouyant force on everyone. You might be wondering how large it is. Calculate the bouyant force on a person. Express your answer in terms of the person's weight. That is, is the bouyant force 1/100 of your weight, 1/1000 of your weight, ...? Assume that a person's density is that of water, which is $1000 \text{ Kg}/\text{M}^3$.

The bouyant force is the weight of the air displaced. Let the volume of the person be V . Then the weight of the air displaced is

$$F_{\text{bouyant}} = \rho_{\text{air}} V g \quad (7)$$

Since the weight of the person is $W = \rho_{\text{water}} V g$, we have

$$\begin{aligned} \frac{F_{\text{bouyant}}}{\text{Person's Weight}} &= \frac{\rho_{\text{air}} V g}{\rho_{\text{water}} V g} \\ &= \frac{\rho_{\text{air}}}{\rho_{\text{water}}} \\ &= 1.29/1000 \\ &= 0.00129 \end{aligned}$$

So if a person weighs 200 pounds, the bouyant force on him is 0.258 pounds or around 4 ounces. We see from this example that the bouyant force on an object that is submerged in a fluid equals the objects weight times $\rho_{fluid}/\rho_{object}$.

Exercise 2.1

Consider the three point objects shown in the figure that are colinear and have masses of $3m_0$, $2m_0$ and m_0 . The object of mass $2m_0$ lies between the other two and is a distance of a from the $3m_0$ mass and $2a$ from the m_0 mass. What is the net gravitational force on the $2m_0$ mass due to the other two? Express your answer in terms of G , a and m_0 .

To find the net force on the $2m_0$ mass we can use Newton's law of gravity for point objects and the superposition principle. The net force on the $2m_0$ mass is the force on it due to the $3m_0$ mass plus the force on it due to the object of mass m_0 .

The force on the $2m_0$ mass due to the object of mass $3m_0$ is

$$\begin{aligned} F_{23} &= G \frac{(2m_0)(3m_0)}{a^2} \\ &= G \frac{6m_0^2}{a^2} \end{aligned}$$

to the left. If we let \hat{i} point in the "+" direction, then we can write:

$$\vec{F}_{23} = -G \frac{6m_0^2}{a^2} \hat{i} \tag{8}$$

The force on the $2m_0$ mass due to the object of mass m_0 is

$$\begin{aligned} F_{21} &= G \frac{(2m_0)(m_0)}{(2a)^2} \\ &= G \frac{m_0^2}{2a^2} \end{aligned}$$

to the right. In terms of \hat{i} we can write

$$\vec{F}_{21} = G \frac{m_0^2}{2a^2} \hat{i} \tag{9}$$

Adding these two forces gives the Net Force on the particle of mass $2m_0$:

$$\begin{aligned}\vec{F}_{net} &= -G\frac{6m_0^2}{a^2}\hat{i} + G\frac{m_0^2}{2a^2}\hat{i} \\ &= -\frac{11}{2}G\frac{m_0^2}{a^2}\hat{i}\end{aligned}$$

So the net force on the middle particle due to the other two is of magnitude $(11Gm_0)/(2a^2)$ to the left.

Exercise 2.2

Consider the three point particles shown in Fig. 2.2. What is the net force on the particle located at the origin due to the other two particles? One particle is located on the "y" axis, and the other on the "x" axis.

As in the last exercise, the net force on the particle at the origin equals the vector sum of the individual forces due to the other two particles. Let \vec{F}_{21} be the force on the particle at the origin due to the particle of mass $3m_0$. Since they are point particles, Newton's law of gravitation applies:

$$\begin{aligned}|\vec{F}_{21}| &= G\frac{(3m_0)(m_0)}{a^2} \\ &= G\frac{3m_0^2}{a^2}\end{aligned}$$

Since the gravitational force is always attractive, the force is directed in the $+y$ direction, or $+\hat{j}$:

$$\vec{F}_{21} = G\frac{3m_0^2}{a^2}\hat{j} \tag{10}$$

Let \vec{F}_{23} be the force on the particle at the origin due to the particle of mass $16m_0$. Since they are both point particles, Newton's law of gravitation applies:

$$\begin{aligned}|\vec{F}_{23}| &= G\frac{(16m_0)(m_0)}{(2a)^2} \\ &= G\frac{4m_0^2}{a^2}\end{aligned}$$

2.1

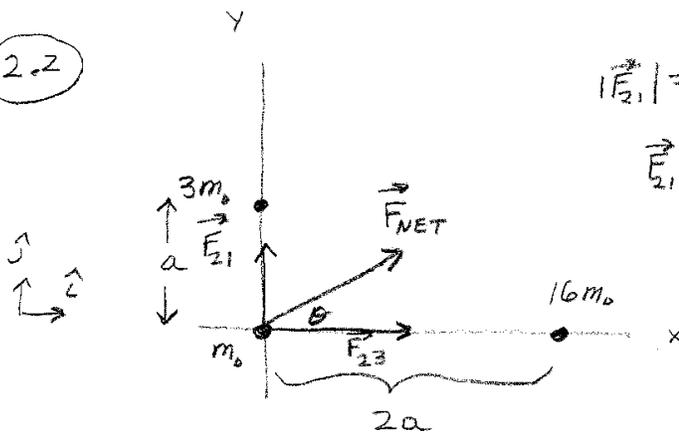


$$|\vec{F}_{23}| = \frac{G(3m_0)(2m_0)}{a^2} = 6 \frac{Gm_0^2}{a^2} \text{ to the left}$$

$$|\vec{F}_{21}| = \frac{G(2m_0)m_0}{(2a)^2} = \frac{Gm_0^2}{2a^2} \text{ to the right}$$

$$|\vec{F}_{NET}| = \frac{6Gm_0^2}{a^2} - \frac{Gm_0^2}{2a^2} = \boxed{\frac{11}{2} \frac{Gm_0^2}{a^2} \text{ to the left}}$$

2.2



$$|\vec{F}_{21}| = \frac{G(3m_0)m_0}{a^2}$$

$$\vec{F}_{21} = 3 \frac{Gm_0^2}{a^2} \hat{j}$$

$$|\vec{F}_{23}| = \frac{G(16m_0)m_0}{(2a)^2} = 4 \frac{Gm_0^2}{a^2}$$

$$\vec{F}_{23} = 4 \frac{Gm_0^2}{a^2} \hat{i}$$

$$|\vec{F}_{NET}| = 5 \frac{Gm_0^2}{a^2}$$

$$\vec{F}_{NET} = \frac{Gm_0^2}{a^2} (4\hat{i} + 3\hat{j})$$

$$\theta = \tan^{-1} \frac{3}{4} \approx 37^\circ$$

The direction of this force on the particle at the origin is to the right, or the $+\hat{i}$ direction:

$$\vec{F}_{23} = G \frac{4m_0^2}{a^2} \hat{i} \quad (11)$$

The net force on the particle at the origin is the **vector** sum of these two forces:

$$\begin{aligned} \vec{F}_{net} &= \vec{F}_{21} + \vec{F}_{23} \\ &= G \frac{3m_0^2}{a^2} \hat{j} + G \frac{4m_0^2}{a^2} \hat{i} \\ &= G \frac{m_0^2}{a^2} (4\hat{i} + 3\hat{j}) \end{aligned}$$

Thus, the magnitude of the net force equals

$$|\vec{F}_{net}| = 5G \frac{m_0^2}{a^2} \quad (12)$$

in a direction 37° from the x-axis.

Exercise 2.3

How much energy in Joules would a 70 Kg person standing on the surface of the earth need to escape the earth?

The Gravitational potential energy U of a point mass and a uniform sphere is

$$U = -G \frac{m_1 m_2}{d} \quad (13)$$

where m_1 and m_2 are the masses of the objects and d is the distance from the center of the sphere to the point mass. Since the mass of the earth is 5.98×10^{24} , and its radius is 6.37×10^6 meters, we have

$$U = -6.67 \times 10^{-11} \frac{(70)(5.98 \times 10^{24})}{(6.37 \times 10^6)} \quad (14)$$

$$= -4.38 \times 10^9 \text{ Joules} \quad (15)$$

Wow, 4.38 billion Joules is a lot of energy. I guess we had better be happy staying here on earth. Why is the potential energy negative? This is because the formula

above for U is calculated using $U = 0$ at $r = \infty$. So if you are infinitely far away from earth, your potential energy is zero. Since the force of gravity is attractive, the potential energy at the surface of the earth is less than at infinity, i.e. less than zero.

Exercise 2.4

Suppose you wanted to leave the earth. What speed would you have to jump up with to escape to infinity?

This escape speed is called the escape velocity, and is the same for an object of any mass. Let the speed that you jump off the surface of the earth be labelled as v_e . Thus, the total energy you have just after you jump is

$$\begin{aligned} E_{tot} &= K.E. + U \\ &= \frac{m}{2}v_e^2 - G\frac{M_e m}{R_e} \end{aligned}$$

where m is your mass, M_e is the mass of the earth, and R_e is the radius of the earth. As you float out to infinity, your total energy remains constant. Thus, when you are a distance r away from the center of the earth, your total energy $E_{tot} = (m/2)v^2 - GM_e m/r$ must be the same as when you were on the earth's surface:

$$\frac{m}{2}v^2 - G\frac{M_e m}{r} = \frac{m}{2}v_e^2 - G\frac{M_e m}{R_e} \quad (16)$$

where v is your speed at a distance r from the earth's center. If you want to reach infinity, $r = \infty$, with zero speed, $v = 0$, then your total energy needs to be zero:

$$\begin{aligned} \frac{m}{2}v_e^2 &= G\frac{M_e m}{R_e} \\ v_e &= \sqrt{\frac{2GM_e}{R_e}} \end{aligned}$$

Substituting values for the earth's mass and radius yields a speed of around 11,000 m/s. If your speed is greater than $\sqrt{2GM_e/R_e}$, then you will have some speed as $r \rightarrow \infty$. Note that the mass m canceled out in the above equation. This means that the escape velocity is the same for all objects.

Exercise 2.5

You are on a spherical planet that has a constant density ρ . The radius of the planet is R . What is the acceleration due to gravity at the surface of the planet? Express your answer in terms of G , R , and ρ .

If an object is right near the surface of the planet, it is approximately a distance R from the center. Then, the gravitational force between the object and the planet is $F_{gravity} = GmM/R^2$, where m is the mass of the object, and M is the mass of the planet. In the Newtonian picture, this is the force that accelerates the object and is equal to ma . Since we are calling this acceleration the acceleration due to gravity and labeling it g , we have:

$$\begin{aligned}mg &= G\frac{mM}{R^2} \\g &= \frac{GM}{R^2}\end{aligned}$$

We can express the mass in terms of the density of the object since mass equals density times volume. For a sphere we have $M = \rho(4/3)\pi R^3$. Substituting into the equation for the mass M gives:

$$\begin{aligned}g &= \frac{G}{R^2}(\rho\frac{4}{3}\pi R^3) \\g &= \frac{4}{3}G\rho\pi R\end{aligned}$$

The larger a planet is, the more the gravitational acceleration at the surface.

Exercise 2.6

Consider a binary star system, which consists of two stars of equal mass, M . The stars are rotating about each other at a distance of d apart. Find an expression for the period T of rotation in terms of G , M , and d . See the figure for a diagram of the motion.

Since the stars are of equal mass, they rotate about a point that is half way between them. That is, each is rotating in a circle of radius $d/2$. Let v be the speed of each star as it moves in circular motion. To travel in a circle with speed v and radius $d/2$, the net force on the star must be directed towards the center of the circle and have

a magnitude of $Mv^2/(d/2)$. The source of this force is the gravitational attractive force between the two stars, GM^2/d^2 . Equating these two quantities yields:

$$G \frac{M^2}{d^2} = \frac{Mv^2}{(d/2)} \quad (17)$$

Note that on the left side we have d^2 , since d is the distance between the two stars. The distance between the stars determines the gravitational force. On the right side of the equation, we use $d/2$ since $d/2$ is the radius of the circle that the stars travel in. Solving for v gives

$$v = \sqrt{\frac{GM}{2d}} \quad (18)$$

To find the period of the motion, we use the fact that the period T equals the circumference divided by the speed:

$$\begin{aligned} T &= \frac{2\pi(d/2)}{v} \\ &= \pi d \sqrt{\frac{2d}{GM}} \end{aligned}$$

Exercise 2.7

Suppose there is a thin ring of mass M and radius a that is fixed in space. A small point object of mass m can move on the axis of the ring as shown in the figure. What is the gravitational potential energy of the small mass m as a function of distance x from the center of the ring?

The gravitational potential energy of two small "point" objects equals $-G$ times the product of their masses divided by the distance between them if we take the potential at infinity to be zero. However, for our problem the ring is not a point mass, so we can not use this formula directly. The ring is a continuous mass distribution, so to solve the problem we need to divide the ring up into small pieces. Then, we can use the formula for two point objects and add up the contributions around the ring. That is, we need to integrate around the ring. This might appear difficult, but it is easy for the case of the ring. **Each part of the ring is the same distance from the location at x .**

Lets divide the ring up into small pieces, each having a mass of ΔM . The gravitational potential energy, ΔU between the point mass m and the piece on the ring of mass ΔM is:

$$\Delta U = -G \frac{m \Delta M}{\sqrt{x^2 + a^2}} \quad (19)$$

Note that we need to use the distance between m and ΔM in the denominator, which is determined from Pythagorus' Theorem. To find the total gravitational potential energy we need to integrate ΔU around the ring.

$$\begin{aligned} U &= \int -G \frac{m \Delta M}{\sqrt{x^2 + a^2}} \\ &= -G \frac{m}{\sqrt{x^2 + a^2}} \int \Delta M \\ U &= -G \frac{mM}{\sqrt{x^2 + a^2}} \end{aligned}$$

You might be wondering why we did not need to use the sin or cos of an angle as we did in lecture for the force between these two objects. This is because the gravitational potential is a scalar. It does not have a direction. Often it is easier to determine the potential than the force between two objects. One can obtain the force between the ring and point mass m from U . The "x-component" of the force is related to the potential by:

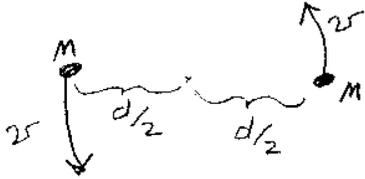
$$F_x = -\frac{\partial U}{\partial x} \quad (20)$$

If we differentiate our expression for U with respect to x , we have

$$\begin{aligned} F_x &= -\frac{\partial U}{\partial x} \\ &= GMm \left(-\frac{1}{2}\right) (x^2 + a^2)^{-3/2} (2x) \\ &= -\frac{GMmx}{(x^2 + a^2)^{3/2}} \end{aligned}$$

which is the same result we obtained in lecture by directly integrating around the ring to obtain the force.

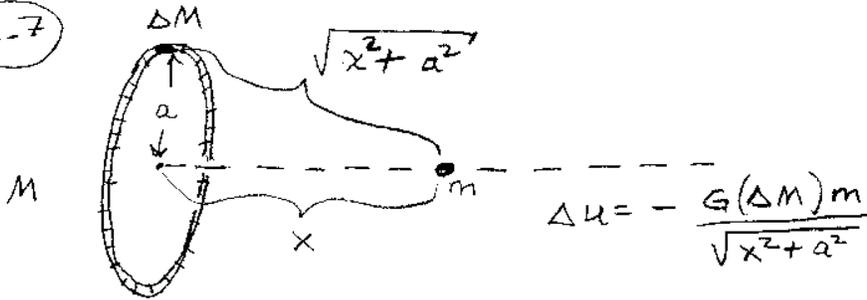
2.6



$$\frac{Mv^2}{d/2} = G \frac{MM}{d^2}$$

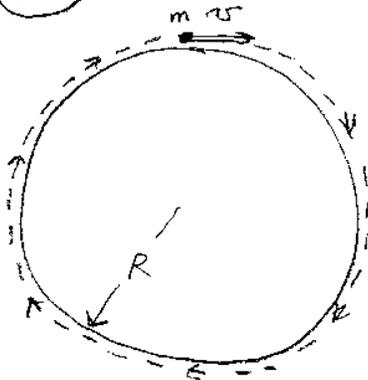
$$v = \sqrt{\frac{GM}{2d}}$$

2.7



$$U = \int \Delta U = \frac{-Gm}{\sqrt{x^2+a^2}} \int \Delta M = \frac{-GmM}{\sqrt{x^2+a^2}}$$

2.8



$$\frac{mv^2}{R} = G \frac{mM}{R^2}$$

$$v = \sqrt{\frac{GM}{R}}$$

but since $g = \frac{GM}{R^2}$

$$v = \sqrt{gR}$$

Exercise 2.8

You are all alone on a spherical planet that has a mass M and a radius R . You are bored, and want to play catch with yourself. You want to throw a baseball so that it orbits just above the surface of the planet and travels all the way around. Then you can catch the ball when it comes back to you. What is the minimum speed that you need to throw the ball with, and how long will it take to complete the orbit? Neglect any friction due to the atmosphere.

For the baseball to travel in circular motion with speed v , radius R , the net force on it must be mv^2/R , where m is the mass of the baseball. Since this force is supplied by the gravitational force, we have

$$\frac{mv^2}{R} = G\frac{mM}{R^2} \quad (21)$$

Notice that the mass of the baseball, m , cancels out, so the speed is the same for any object. Solving for v we have

$$v = \sqrt{\frac{GM}{R}} = \sqrt{gR} \quad (22)$$

where g is the acceleration due to gravity. For earth, this is a speed of $v \approx 7900$ m/s. The time for one complete orbit can be found by dividing the circumference by this speed:

$$T = \frac{2\pi R}{\sqrt{gR}} = 2\pi\sqrt{\frac{R}{g}} \quad (23)$$

For earth, the period becomes $T \approx 5066$ seconds, or around 84 minutes. This will give you plenty of time to turn around and catch the ball.

Exercise 2.9

You feel an attraction to the student sitting next to you in lecture. You think that it might be due to the gravitational force. You estimate the student's mass to be 50 Kg, and you know your mass to be 70 Kg. The distance between you and the student is 1 meter. What is the attractive gravitational force between you and the student?

We can approximate the gravitational force between you and the student next to you by assuming that you are both point objects. Then we can use Newton's formula for point objects:

$$\begin{aligned}
|\vec{F}| &= G \frac{m_1 m_2}{r^2} \\
&= (6.67 \times 10^{-11}) \frac{(70)(50)}{1^2} \\
&\approx 2.33 \times 10^{-7} \text{ Newtons}
\end{aligned}$$

This is a very small force, which you cannot even feel. So, any attraction you might feel towards the student sitting next to you is not gravitational. It might have something to do with biology.

Exercise 2.10

A small object of mass m is located at the center of a thin semi-circular rod. The rod is in the shape of a semi-circle of radius R and has a total mass M . What is the gravitational force that the object of mass m feels due to the thin semi-circular rod?

From symmetry, the mass m will be attracted in the $+x$ direction (towards the middle of the rod). The rod is not a point object, so we need to divide it up into small pieces and add up the force vector contributions from each piece. Lets divide the rod into N equal (small) pieces, each one having a mass $\Delta M = M/N$ as shown in the figure. The magnitude of the attractive gravitational force between the mass m and the piece ΔM is given by

$$|\Delta \vec{F}| = G \frac{m \Delta M}{R^2} \quad (24)$$

We can use this formula because both m and ΔM are **point** objects. Note that the direction of the force that m feels from the piece ΔM is towards the piece ΔM . As we integrate around the rod, the variable that is changing is the angle between ΔM and m , which we label as θ in the figure. Let $R\Delta\theta$ be the arc length of the mass ΔM . That is, θ is the angle that ΔM subtends as shown in the figure. In order to add up the $\Delta \vec{F}$ from the pieces, we need to relate ΔM to $\Delta\theta$. We can do this by setting up a proportion:

$$\begin{aligned}
\frac{\Delta M}{M} &= \frac{\Delta\theta}{\pi} \\
\Delta M &= \frac{M}{\pi} \Delta\theta
\end{aligned}$$

Thus the magnitude of the force that m feels from the piece of mass ΔM is

$$|\Delta \vec{F}| = G \frac{mM}{\pi R^2} \Delta\theta \quad (25)$$

When we integrate around the semi-circle, the net force will be in the $+x$ direction. Thus, we only need to add up the x-component of $\Delta \vec{F}$. The x-component F_x is equal to $F_x = |\Delta \vec{F}| \cos\theta$. The x-component of the force that m feels from ΔM is

$$\Delta F_x = G \frac{mM}{\pi R^2} \cos\theta \Delta\theta \quad (26)$$

Adding up the ΔF_x contributions around the semi-circle yields the following integral:

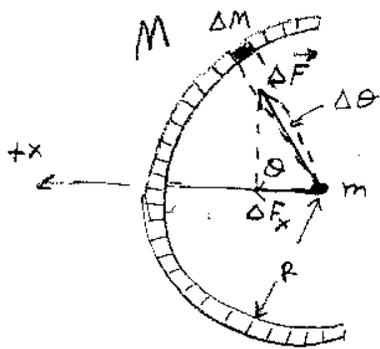
$$F_x = G \frac{mM}{\pi R^2} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta \quad (27)$$

The limits on the integral are from $-\pi/2$ to $\pi/2$ because θ varies between these values as we move around the semi-circle. To finish the problem, we need to do some math. Since $\int \cos\theta d\theta = \sin\theta$ and $\sin(\pi/2) - \sin(-\pi/2) = 2$, we have

$$F_x = G \frac{2mM}{\pi R^2} \quad (28)$$

Note that the semi-circular rod feels the same force towards m .

2-10



$$|\vec{\Delta F}| = G \frac{m \Delta M}{R^2}$$

$$\frac{\Delta M}{M} = \frac{\Delta \theta}{\pi}$$

$$\Delta M = \frac{M}{\pi} \Delta \theta$$

$$|\vec{\Delta F}| = \frac{GMm}{R^2 \pi} \Delta \theta$$

$$\Delta F_x = |\vec{\Delta F}| \cos \theta$$

$$= \frac{GMm}{R^2 \pi} \cos \theta \Delta \theta$$

$$F_x = \sum \Delta F_x = \int_{-\pi/2}^{\pi/2} \frac{GMm}{R^2 \pi} \cos \theta d\theta$$

$$F_x = \frac{GMm}{R^2 \pi} \sin \theta \Big|_{-\pi/2}^{\pi/2} = \boxed{\frac{2GMm}{R^2 \pi}}$$