Exercises on Statics and Rotational Dynamics

Exercise 1.1
Jack, mass 50, Kg and Jill, mass 30 Kg, are sitting on a massless teeter-totter. Jack is 3 meters to the left of the fulcrum. How far away is Jill from the fulcrum?

Since the teeter-totter is not rotating (or moving), the net torque about any point equals zero. Let's take the fulcrum as the point to evaluate the torque. Jill's weight produces a clockwise torque about the fulcrum equal to $30gx$, where g is the acceleration due to gravity. Jack's weight produces a counter-clockwise torque about the fulcrum equal to $50(3)g$ about the fulcrum. Since the net torque must be zero, the clockwise and counter-clockwise torques must balance:

$$\begin{aligned} 3(50) &= 30x \\ x &= \frac{150}{30} \\ x &= 5\, meters \end{aligned}$$

Note that the calculation of the torques are easy in this case, since the force is perpendicular to the radius vector.

Exercise 1.2
A box, of mass 40 Kg, is placed at the end of a uniform plank. The plank has a mass of 80 Kg and a length of 10 meters. Where should the plank (plus box) be placed so that it balances on sharp fulcrum? That is, what is x in the figure?

Since the plank (plus box) is stationary, the net torque about any axis must be zero. Let's choose the fulcrum as the point to evaluate the net torque. The box produces a counter-clockwise torque about the fulcrum of magnitude $40x$. We can treat the plank as if all the mass is located at the center, that is a mass of 80 Kg located at the center. Since the center of the plank is $5 - x$ meters from the fulcrum, the plank produces a clockwise torque about the fulcrum of magnitude $80(5-x)$. Since the net torque must be zero, the clockwise and counter-clockwise torques balance:

$$\begin{aligned} 40x &= 80(5-x) \\ 40x &= 400 - 80x \\ x &= 3.333\, meters \end{aligned}$$

As in the last exercise, the torques were easy to calculate since the forces are perpendicular to the radius vectors.

Exercise 1.3
Consider the following two vectors:

$\vec{A}$: 5 units at 37° North of East

$\vec{B}$: 10 units at 37° North of West

Find $\vec{A} \times \vec{B}$.

There are two ways to calculate the cross product: by using the unit vectors, or as $|\vec{A}||\vec{B}|sin\theta$. In the latter case, the direction is determined using the right hand rule. In terms of the unit vectors, $\vec{A} = 4\hat{\imath} + 3\hat{\jmath}$, and $\vec{B} = -8\hat{\imath} + 6\hat{\jmath}$. The cross product can be calculated using the determinant as shown in the figures page or

$$
\begin{aligned}
\vec{A} \times \vec{B} &= |\vec{A}||\vec{B}|sin\theta \\
&= 5(10)sin(106°)\hat{k} \\
&= 48\hat{k}
\end{aligned}
$$

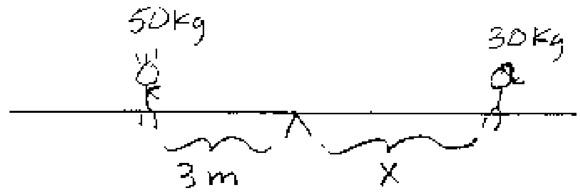
where $\hat{k}$ is a unit vector in the "z-direction". Note that $\vec{B} \times \vec{A} = -48\hat{k}$.

Exercise 1.4
You want to build a sign that will hang at the end of a long rod. The sign has a weight of 50 pounds. The rod has a weight of 20 pounds and a length of 8 feet. One end of the rod is attached to the wall, and the other end is held up by a rope. The tension in the rope is labeled $\vec{T}$ in the figure. Find the tension T in the rope and the force $\vec{F}$ that the wall exerts on the rod.

Consider the forces on the rod. If the rod is in static equilibrium, then two conditions must be met: **The sum of the forces on the rod must add up to zero**, and **the net torque about any axis must also be zero**.

Let's start with the torque requirement first. We are free to choose any axis as our axis for calculating the torque. Let's choose the point "O" in the figure, which is where the force $\vec{F}$ acts. The net torque about the point "O" is given by

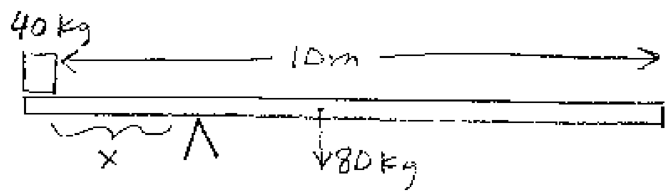
(1)



$$3(50) = x(30)$$

$$\boxed{x = 5m}$$

(2)



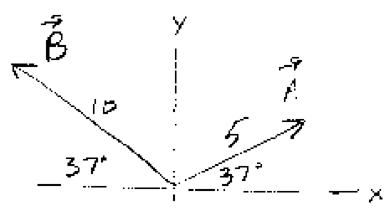
$$40x = 80(5 - x)$$

$$40x = 400 - 80x$$

$$120x = 400$$

$$x = 3.3\bar{3}\ m$$

(3)



$$\vec{A} = 5\cos 37\ \hat{i} + 5\sin 37\ \hat{j}$$

$$\vec{A} = 4\hat{i} + 3\hat{j}$$

$$\vec{B} = -8\hat{i} + 6\hat{j}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 3 & 0 \\ -8 & 6 & 0 \end{vmatrix} = \hat{i}(3\cdot 0 - 0\cdot 6) + \hat{j}((-4)\cdot 0 - (-8)(0)) + \hat{k}(4(6) - 3(-8)) = \boxed{48\hat{k}}$$

or $\vec{A}\times\vec{B} = 5(10)\sin(106^\circ) \approx 48\ \hat{k}$

$$Net\ torque\ about\ O = 0(F) - 4(20) - 8(50) + 8T sin(37°) \tag{1}$$

The force $\vec{F}$ does not produce any torque (twisting force) about the point "O" because it acts at this point. (i.e. r = 0). The 20 pound force of the rod and the 50 pound force of the sign produce a clockwise twist about "O", so the signs are negative. The tension T produces a counter-clockwise twist and has a positive sign. Since the Net Torque must be zero, we have:

$$\begin{aligned} 0 &= -4(20) - 8(50) + 8T sin(37°) \\ T &= 100\ pounds \end{aligned}$$

To find the force $\vec{F}$, we use the condition that the net force on the rod equals zero. It is easiest to express all the forces in terms of their components, or equivalently in terms of the unit vectors. Using $\hat{i}$ and $\hat{j}$ as defined in the figure, the vector $\vec{T}$ is equal to $\vec{T} = -100cos(37°)\hat{i} + 100sin(37°)\hat{j}$. Thus, $\vec{T} = -80\hat{i} + 60\hat{j}$ pounds. Setting the sum of the forces equal to zero gives:

$$\begin{aligned} 0 &= \vec{F} - 20\hat{j} - 50\hat{j} + (-80\hat{i} + 60\hat{j}) \\ \vec{F} &= (80\hat{i} + 10\hat{j})\ pounds \end{aligned}$$

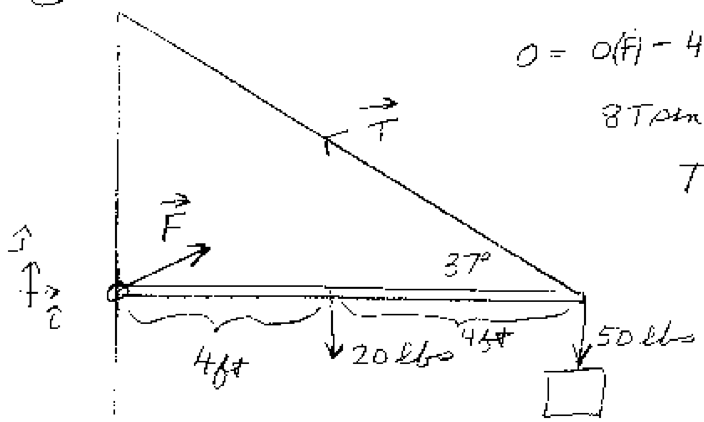
The magnitude of $\vec{F}$ equals $|\vec{F}| = \sqrt{80^2 + 10^2} \approx 80.6\ pounds$, and is directed at an angle $\theta = tan^{-1}(10/80) \approx 7.1°$ north of east. Note that the force $\vec{F}$ is not directed along the rod.

Exercise 1.5

Brandon needs to climb up a massless ladder that is against a wall. Brandon's mass is m, the ladder has a length L, and makes an angle of θ with the ground. The coefficient of static friction between the ladder and the ground is μ, and there is no friction between the ladder and the wall. How far up the ladder, the distance d in the figure, can Brandon climb before the bottom of the ladder slips and Brandon crashes to the ground?

We first need to identify all the forces on the ladder. Since the ladder is massless, there is no downward force of gravity located at its center of mass. Brandon exerts a force of mg located a distance d from the base of the ladder. The ground exerts a

(4)



$$0 = 0(F) - 4(20) - 8(50) + 8T\sin 37^\circ$$

$$8T\sin 37 = 480$$

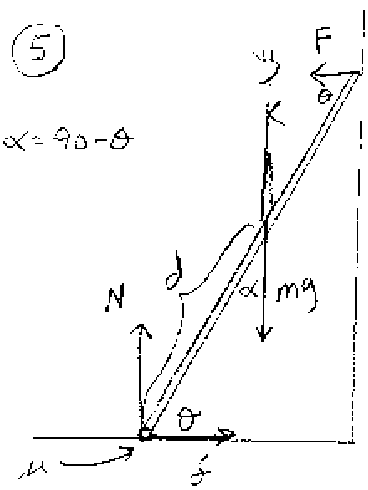
$$T = \frac{480}{8\sin 37} = \boxed{100\text{ lbs}}$$

$$\vec{F} - 20\hat{j} - 50\hat{j} + \underbrace{(-80\hat{i} + 60\hat{j})}_{\vec{T}} = 0$$

$$\boxed{\vec{F} = 80\hat{i} + 10\hat{j} \text{ pounds}}$$

(5)

$\alpha = 90 - \theta$



$f_{max} = \mu N$

$\Sigma \vec{F}_i = 0 \Rightarrow$ $N = mg$, $F = f$

$\Sigma \vec{\tau} = 0$

$$-mgd\sin\alpha + FL\sin\theta = 0$$

$$mgd\cos\theta = FL\sin\theta$$

$$mgd = fL\tan\theta$$

$$mgd_{max} = f_{max}L\tan\theta$$

$$mgd_{max} = \mu mgL\tan\theta$$

$$\boxed{d_{max} = \mu L\tan\theta}$$

force on the ladder, which we break up into two "components". One component is perpendicular to the ground, which we call $\vec{N}$, and the other is parallel to the ground and is the force of friction, $\vec{f}$. The last force is the force that the wall exerts on the ladder, labeled $\vec{F}$ in the figure. Since there is no friction between the ladder and the wall, $\vec{F}$ must be perpendicular to the wall.

It is easiest to first satisfy the condition that the net force on the ladder equals zero. In the vertical direction N must equal mg: $N = mg$. In the horizontal direction F must equal f: $F = f$. Now for the torques. It is easiest to choose the base of the ladder as our axis, since there are two components that act there. With this choice:

$$Torque\ about\ ladder\ base = 0(N) + 0(f) - mgd\ sin\alpha + FL\ sin\theta$$

where the angle $\alpha = 90^\circ - \theta$ is the angle between $m\vec{g}$ and the ladder. N and f are multiplied by zero, since they act at the location of the axis. Since the net torque must be zero, we have:

$$\begin{aligned} 0 &= -mgd\ cos\theta + fL\ sin\theta \\ mgd\ cos\theta &= fL\ sin\theta \\ mgd &= fL\ tan\theta \end{aligned}$$

The maximum value for d is determined by the maximum value for the frictional force f. Since $f_{max} = \mu N = \mu mg$ we have

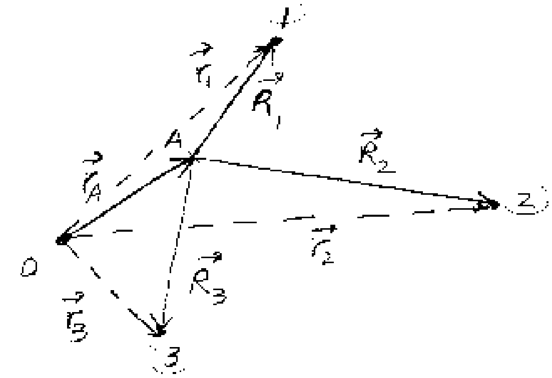
$$\begin{aligned} mgd_{max} &= f_{max}L\ tan\theta \\ mgd_{max} &= \mu mgL\ tan\theta \\ d_{max} &= \mu L\ tan\theta \end{aligned}$$

Note that if $\mu\ tan\theta$ is greater than one, then $d_{max} > L$ and Brandon can climb up to the top of the ladder without it slipping.

Exercise 1.6

Prove the following statement: If the sum of the forces equals zero and the net torque about one axis equals zero, then the net torque about any axis is also equal to zero.

(6)

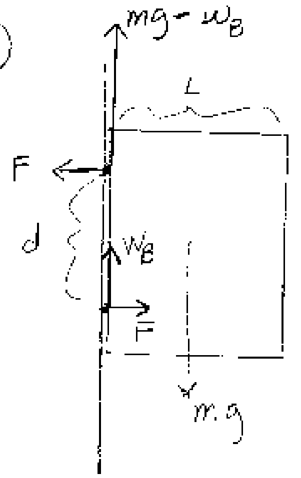


$$\vec{r}_i = \vec{r}_A + \vec{R}_i$$

$$\sum \vec{r}_i \times \vec{F}_i = \sum (\vec{r}_A + \vec{R}_i) \times \vec{F}_i = \sum(\vec{r}_A \times \vec{F}_i) + \sum(\vec{R}_i \times \vec{F}_i)$$

$$= \vec{r}_A \times \underbrace{\sum \vec{F}_i}_{=0} + \underbrace{\sum \vec{R}_i \times \vec{F}_i}_{=0} = 0$$

(7)



$$0 = -mg\frac{L}{2} + Fd$$

$$\boxed{F = \frac{mgL}{2d}}$$

Let's prove the statement for three forces, which can be generalized to any number of forces. Suppose three forces act on an object at three different locations. We label the forces as $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$. If the sum of the forces are equal to zero, this means that

$$\begin{aligned} \vec{F}_1 + \vec{F}_2 + \vec{F}_3 &= 0 \\ \sum_{i=1}^{3} \vec{F}_i &= 0 \end{aligned}$$

Let the net torque about the point "A" equal zero. If we let $\vec{R}_i$ be the displacement vector from "A" to the location of force "i", then the net torque about "A" equals

$$\begin{aligned} \vec{\tau}_{net}(A) &= \vec{R}_1 \times \vec{F}_1 + \vec{R}_2 \times \vec{F}_2 + \vec{R}_3 \times \vec{F}_3 \\ \vec{\tau}_{net}(A) &= \sum_{i=1}^{3} \vec{R}_i \times \vec{F}_i \end{aligned}$$

Let "O" be another point. Let $\vec{r}_i$ be the displacement vector from "O" to the location of force "i". The torque about the point "O" is

$$\vec{\tau}_{net}(O) = \sum_{i=1}^{3} \vec{r}_i \times \vec{F}_i$$

If we let $\vec{r}_A$ be the vector from "O" to "A", then

$$\vec{r}_i = \vec{r}_A + \vec{R}_i$$

Substituting this relationship into the torque equation gives:

$$\begin{aligned} \vec{\tau}_{net}(O) &= \sum_{i=1}^{3} (\vec{r}_A + \vec{R}_i) \times \vec{F}_i \\ \vec{\tau}_{net}(O) &= \sum_{i=1}^{3} \vec{r}_A \times \vec{F}_i + \sum_{i=1}^{3} (\vec{R}_i \times \vec{F}_i) \\ \vec{\tau}_{net}(O) &= \vec{r}_A \times \sum_{i=1}^{3} \vec{F}_i + \sum_{i=1}^{3} (\vec{R}_i \times \vec{F}_i) \\ \vec{\tau}_{net}(O) &= \vec{r}_A \times 0 + 0 \\ \vec{\tau}_{net}(O) &= 0 \end{aligned}$$

The first zero is because the sum of the forces equals zero, $(\sum \vec{F}_i = 0)$, and the second zero is because the sum of the torques about "A" equals zero.

This is a nice result. If the sum of the forces equals zero, then we only need to require that the sum of the torques about one point equals zero. If this is satisfied, then we know that the sum of the torques equals zero about any point we choose.

Exercise 1.7
Dave wants to hang a door that has a mass m and a width L. The door will connect to the wall at two hinges, which are separated by a distance d as shown in the figure. What are the maximum forces that the top hinge must exert on the door?

Lets identify all the forces on the door: The weight of the door, which can be considered as acting at the center of gravity of the door; and the forces at each hinge. The force at each hinge can be broken up into a vertical component and a horizontal component. Since the sum of the forces must add up to zero, the sum of the vertical components that the hinges exert must add up to mg. If we let W_B be the vertical force that the bottom hinge exerts on the door, then $mg - W_B$ is the vertical force that the top hinge exerts on the door. (The sum of these two forces must be mg). Since the only horizontal forces exerted on the door are due to the hinges, the horizontal force for the top hinge must be opposite in direction and equal in magnitude to the horizontal force for the bottom hinge. We label this force F in the figure.

Now for the net torque. If we choose as our axis the location of the top hinge, we have

$$Torque\ about\ top\ hinge = -mg\frac{L}{2} + Fd$$

Note that the torque about the top hinge due to W_B is zero, since $\vec{W}_B$ points directly at the top hinge. From this equation we can solve for the horizontal forces on the hinges:

$$F = \frac{mgL}{2d}$$

The largest that the vertical components can be is mg.

Exercise 2.1
What is the angular velocity vector of the earth?

The angular velocity equals the number of radians/time, so

$$\begin{aligned} \omega &= \frac{2\pi \ radians}{one \ day} \\ &= \frac{2\pi}{24(3600) \ sec} \\ &\approx 7.27 \times 10^{-5} \ rad/sec \end{aligned}$$

This is the magnitude of the angular velocity, but what is the direction? Since the sun sets in the west, the direction of $\vec{\omega}$ is from the south to the north pole.

Exercise 2.2
What is the angular acceleration of the earth about its axis?

The angular acceleration is the change in the angular velocity,

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

Since the earth's angular velocity vector does not change hardly at all (each day has nearly the same duration), the angular acceleration of the earth is zero. Actually, $\vec{\omega}$ will change very slightly due to atmospheric changes, but $\vec{\alpha}$ is essentially zero.

Exercise 2.3
The latitude of Los Angeles is around 34° North of the equator. What is the speed of a person in Los Angeles due to the earth's rotation about its axis?

We can solve this by using $\vec{v} = \vec{r} \times \vec{\omega}$. $\vec{\omega}$ points towards the north pole. The angle between $\vec{r}$ and the north pole is equal to $90^\circ - 34^\circ = 56^\circ$. Since the radius of the earth is approximately 6.37×10^6 meters,

$$\begin{aligned} |\vec{v}| &= |\vec{r}||\vec{\omega}| sin 56^\circ \\ &\approx (6.37 \times 10^6)(7.27 \times 10^{-5}) sin 56^\circ \\ &\approx 384 \ m/s \end{aligned}$$

Note, that if you are at the north pole, $\theta = 0$, so your speed about the axis is 0.

Exercise 2.4
Bugsy spins the lottery wheel counter-clockwise until it is rotating at 2 revolutions/sec. The wheel is a clockface with 12 equal divisions labeled $1 \rightarrow 12$ going clockwise. When the 12 is at the top, rotating at 2 revolutions/sec, he lets it slow down on its own. It takes 44.2 seconds to slow down. Assuming that the angular acceleration is constant, what two numbers does it land between?

The initial angular velocity is $\omega_0 = 2(2\pi) = 4\pi$ radians/sec. Since it takes 44.2 seconds to slow down, the angular acceleration is $\alpha = 4\pi/44.2 = \pi/11.05 \approx 0.284\ r/s^2$. If the angular acceleration is constant, then the angle that is swept out is

$$\begin{aligned} \theta &= \frac{\alpha}{2}t^2 + \omega_0 t + \theta_0 \\ &= \frac{\pi}{22.1}44.2^2 + 4\pi(44.2) \\ &= (265.2)\pi\ radians \end{aligned}$$

The number of revolutions that the wheel turns before it stops is $(265.2)\pi/(2\pi) = 132.6$ revolutions. So the wheel only completes the last 0.6 of a revolution. Multiplying by 12 gives $0.6(12) = 7.2$. Thus, the wheel stops between 7 and 8.

Exercise 2.5
What is the rotational inertia I_{cm} for a uniform disk about an axis through the center and perpendicular to the plane of the disk? Let the radius of the disk be R and it mass M.

The easiest way to find I_{cm} for the disk is to divide the disk up into concentric rings. Consider a ring of radius r and width Δr. The rotational inertial for this ring, ΔI, about an axis through its center is $\Delta I = \Delta m\ r^2$, where Δm is the mass of the ring. If M is the total mass of the disk, then the mass of the ring is equal to $\Delta m = M\Delta A/(\pi R^2)$ where ΔA is the area of the ring. The area of the ring is given by: $\Delta A = 2\pi r\ \Delta r$, since $2\pi r$ is the circumference of the ring and Δr is its thickness. So we have

$$\Delta I = M\frac{2\pi r\ \Delta r}{\pi R^2}r^2$$

We can add up all the rings by integrating from $r = 0$ to $r = R$:

$$
\begin{aligned}
I_{cm} &= \sum \Delta I \\
&= \int_0^R M \frac{2\pi r}{\pi R^2} r^2 dr \\
&= \frac{2M}{R^2} \int_0^R r^3 dr \\
&= \frac{MR^2}{2}
\end{aligned}
$$

Exercise 2.6
What is the rotational inertia for a thin ring of radius R and mass M for rotation about an axis parallel to the ring and through its center (see the figure)?

We need to integrate around the ring. Divide the ring up into small pieces that subtend an angle of $\Delta\theta$. Consider a piece that is an angle θ from the axis. The distance to the axis is $R\ sin\theta$. Let the mass of the small piece be Δm. Then the contribution to the rotational inertia from this small piece is

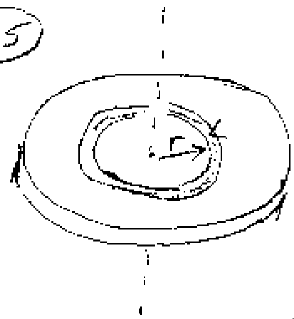
$$\Delta I = \Delta m\ (R\ sin\theta)^2$$

We need to determine Δm in terms of M and $\Delta\theta$. This can be done by using ratios: $(\Delta m/M) = (\Delta\theta/(2\pi))$, or $\Delta m = M\Delta\theta/(2\pi)$. Thus, we have

$$
\begin{aligned}
I_{cm} &= \sum \Delta I \\
&= \sum M \frac{\Delta\theta}{2\pi} R^2 sin^2\theta \\
&= \frac{MR^2}{2\pi} \int_0^{2\pi} sin^2\theta d\theta \\
&= \frac{MR^2}{2}
\end{aligned}
$$

Exercise 2.7
Consider the mass-pulley set-up shown in the figure. The block of mass m_1 is hanging by a massless cord. The cord goes over a pulley and is attached to another block of mass m_2. Mass m_2 can slide without friction on the top of a horizontal table. The

(2.5)



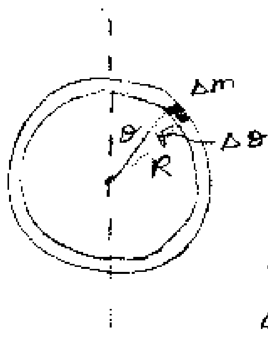
$$\frac{\Delta m}{M} = \frac{2\pi r\,\Delta r}{\pi R^2}$$

$$\Delta m = M\,\frac{2r}{R^2}\,\Delta r$$

$$\Delta I = \Delta m\, r^2 = \frac{M 2 r^3}{R^2}\,\Delta r$$

$$I = \int_0^R \frac{M 2 r^3}{R^2}\,dr = \frac{2M}{R^2}\,\frac{r^4}{4}\Big|_0^R = \frac{MR^2}{2}$$

(2.6)



$$\frac{\Delta m}{M} = \frac{\Delta \theta}{2\pi}$$

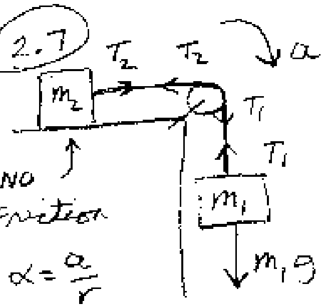
$$\Delta m = M\,\frac{\Delta\theta}{2\pi}$$

$$\Delta I = \Delta m\,(R\sin\theta)^2$$

$$\Delta I = \frac{M}{2\pi}\,\Delta\theta\, R^2 \sin^2\theta$$

$$I = \frac{MR^2}{2\pi}\int_0^{2\pi} \sin^2\theta\, d\theta = \frac{MR^2}{2}$$

(2.7)



$\alpha = \frac{a}{r}$

$$\boxed{T_2 = m_2 a}$$

$$\boxed{m_1 g - T_1 = m_1 a}$$

$$T_1 r - T_2 r = I\alpha = \frac{mr^2}{2}\,\frac{a}{r}$$

$$\boxed{T_1 - T_2 = \frac{m}{2} a}$$

$$m_1 g = \left(m_1 + m_2 + \frac{m}{2}\right) a \Rightarrow \boxed{a = \frac{m_1 g}{m_1 + m_2 + \frac{m}{2}}}$$

pulley is a solid disk with a mass of m and a radius r. What is the acceleration of the system?

To solve this problem, it is easiest to consider each object one at a time. Let T_1 be the tension in the part of the cord attached to m_1, and T_2 the tension in the part of the cord attached to m_2. The net force on m_1 is $m_1 g - T_1$. The net force must be the mass times acceleration:

$$m_1 g - T_1 = m_1 a$$

where a is the acceleration of the system. The only force on m_2 is T_2. So we have

$$T_2 = m_2 a$$

Now we need to determine the net torque on the pulley. The cords apply a force perpendicular to the radius. So T_1 produces a torque equal to rT_1 clockwise, and T_2 produces a torque equal to rT_2 counter-clockwise. Since positive acceleration is a clockwise rotation, the net torque is $rT_1 - rT_2$ which is equal to $I\alpha$:

$$rT_1 - rT_2 = I\alpha$$

The rotational inertial I for a disk rotation about an axis perpendicular to its surface and through its center is $I = mr^2/2$. If the cord does not slip we have $\alpha = a/r$. Substituting these expressions into the equation above, we have

$$\begin{aligned} rT_1 - rT_2 &= I\alpha \\ rT_1 - rT_2 &= \frac{mr^2}{2}\frac{a}{r} \\ T_1 - T_2 &= \frac{m}{2}a \end{aligned}$$

The equations for the three objects are

$$\begin{aligned} m_1 g - T_1 &= m_1 a \\ T_2 &= m_2 a \\ T_1 - T_2 &= \frac{m}{2}a \end{aligned}$$

Adding the right sides and the left sides gives:

$$
\begin{aligned}
m_1 g &= (m_1 + m_2 + \frac{m}{2})a \\
a &= \frac{m_1 g}{m_1 + m_2 + m/2}
\end{aligned}
$$

Exercise 2.8
A meter stick can pivot at one end. The meter stick is held horizontally to the ground, and then let go. It swings down. What is the speed of the tip of the meter stick at the bottom of the swing?

We can use the work-energy theorem to help us solve the problem. The net work is the change in the kinetic energy. The kinetic energy is the rotational energy about the pivot point (or axis of rotation). The only force that does work is the force of gravity, since the force at the pivot point does not "push" the object any distance:

$$
W_g = \frac{I_{end}}{2}\omega_f^2 - \frac{I_{end}}{2}\omega_i^2
$$

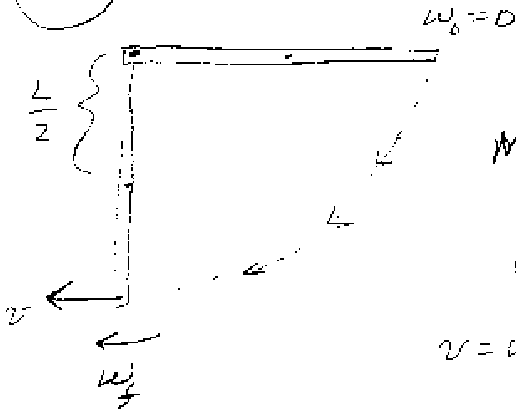
The work done by gravity, W_g, is mg times the change in the height of the center of mass. So $W_g = mgl/2$, where $l = 1\ m$ is the length of the stick. The rotational inertia of a rod with the axis at the end is $I_{end} = ml^2/3$. Thus, we have

$$
\begin{aligned}
mg\frac{l}{2} &= \frac{ml^2/3}{2}\omega_f^2 - 0 \\
\omega_f &= \sqrt{\frac{3g}{l}}
\end{aligned}
$$

The speed of the end of the stick is $v = l\omega$, so the speed at the bottom of the swing is $v = \sqrt{3gl}$.

Exercise 2.9
While in the bathroom, Jerry places a clip of mass m_1 at the end of the roll of toilet paper. The clip falls and the paper unwinds. If we model the toilet paper as a uniform cylinder of radius r and mass m, what is the acceleration of the clip?

(2.8)



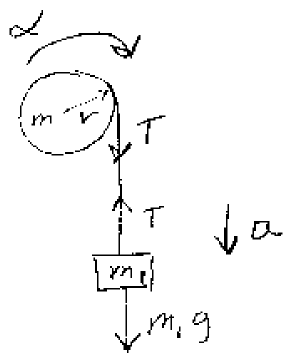
$$W_g = \frac{I}{2}\omega_f^2 - \frac{I}{2}\omega_0^2$$

$$Mg\frac{L}{2} = \frac{\left(\frac{ML^2}{3}\right)}{2}\omega_f^2 - 0$$

$$\omega_f = \sqrt{3g/L}$$

$$v = \omega_f L = \boxed{\sqrt{3gL}}$$

(2.9)



$$\alpha = \frac{a}{r}$$

$$m_1 g - T = m_1 a$$

$$rT = I\alpha$$

$$T = \frac{mr^2}{2r}\left(\frac{a}{r}\right) = \frac{m}{2}a$$

$$m_1 g - \frac{m}{2}a = m_1 a$$

$$\boxed{a = \frac{m_1 g}{m_1 + m/2}}$$

To solve this problem, it is easiest to consider each object separately. Let the tension in the hanging paper be labeled T. Then the net force on the clip is $m_1 g - T$. If we assume that the mass of the hanging paper is negligable, then the roll will feel the same force T pulling it downward. The net torque on the roll of paper is rT. For m_1 we have:

$$m_1 g - T = m_1 a$$

For the paper roll we have:

$$rT = I\alpha$$

The rotational inertia for a cylinder rotating about its center is $I = mr^2/2$. The connection between a and α is $\alpha = a/r$. Substituting into the last equation gives:

$$\begin{aligned} rT &= \frac{mr^2}{2}\frac{a}{r} \\ T &= \frac{m}{2}a \end{aligned}$$

Combining this equation with the first gives

$$\begin{aligned} m_1 g - T &= m_1 a \\ m_1 g - \frac{m}{2}a &= m_1 a \\ a &= \frac{m_1 g}{m_1 + m/2} \end{aligned}$$

Exercise 3.1
A cylindrically symmetric object is rolling without friction down an incline. If the radius of the object is R and the rotational inertial about the center of mass is I_{cm}, what is the acceleration of the rolling object down the incline?

This is an excellent example to apply the following physics: **The acceleration of the center of mass equals the net force; the angular acceleration about the center of mass equals the net torque about the center of mass.**

Lets identify the forces on the object: gravity acting at the center of mass, the normal force from the incline and the frictional force from the incline. (Actually the last two forces are really the one force on the object from the incline). We can break up the force of gravity, $m\vec{g}$, into a component parallel to the plane ($mgsin\theta$) and one perpendicular to the plane ($mgcos\theta$). Since the object stays on the plane, $N = mgcos\theta$. The net force down the plane equals $mg - f$, where f is the frictional force. The net force equals the mass times the acceleration of the center of mass:

$$mg\ sin\theta - f = ma$$

The net torque about the center of mass equals the angular acceleration about the center of mass. The net torque about the center of mass due to $m\vec{g}$ equals zero since it acts at the center, and the net torque due to $\vec{N}$ equals zero since $\vec{N}$ points towards the center of mass. The only force that gives non-zero torque is the frictional force f. Since the moment arm equals r (the radius of the object), we have

$$rf = I_{cm}\alpha$$

where I_{cm} is the rotational inertia about the center of mass. If the object rolls without slipping, $\alpha = a/r$. Thus, $f = (I_{cm}\alpha)/r^2$. Substituting this expression for f into the first equation gives:

$$mg\ sin\theta - \frac{I_{cm}}{r^2}a = ma$$

Solving for a yields:

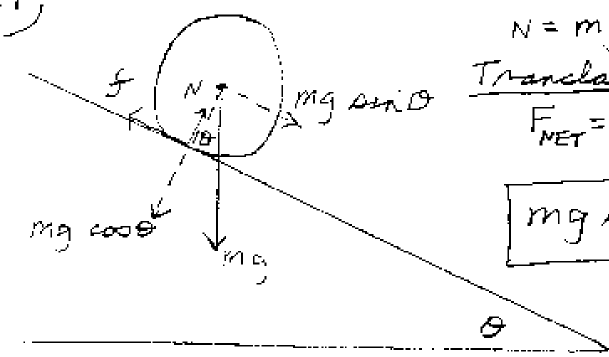
$$a = \frac{g\ sin\theta}{1 + \frac{I_{cm}}{mr^2}}$$

From this expression, one can see that the critical quantity for determining the acceleration is $I_{cm}/(mr^2)$, which is unitless. For a loop this is 1, for a cylinder it is 1/2, and for a sphere it is 2/5.

Exercise 3.2

Consider the cylindrical object of exercise 3.1. What is the maximum angle that the incline can have so that the object does not slip?

(3.1)



$N = mg\cos\theta$

Translation down incline

$F_{NET} = mg\sin\theta - f = ma$

$$\boxed{mg\sin\theta - f = ma}$$

Rotation about the c.m.

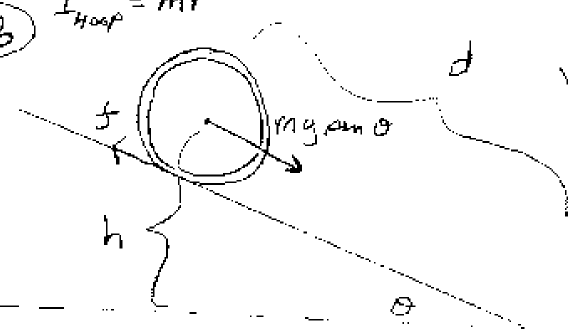
$$\boxed{fr = I_{cm}\alpha}$$

$f = \frac{I_{cm}\alpha}{r} = \frac{I_{cm}a}{r^2}$

$mg\sin\theta - \frac{I_{cm}}{r^2}a = ma$

$$\boxed{a = \frac{g\sin\theta}{1 + \frac{I_{cm}}{mr^2}}}$$

(3.3) $I_{Hoop} = mr^2$



$W_{NET} = (mg\sin\theta - f)\,d$

$+ fd$

work for rotational energy

$W_{NET} = mg\sin\theta\, d - fd + fd = mgh = \frac{m}{2}v^2 + \frac{I}{2}\omega^2$

$mgh = \frac{m}{2}v^2 + \frac{(mr^2)}{2}\left(\frac{v}{r}\right)^2 = mv^2$ $\quad \boxed{v = \sqrt{gh}}$

If friction is not large enough, then the object cannot rotate fast enough to keep up with the translational acceleration. $f_{max} = \mu mgcos\theta$, where μ is the coefficient of static friction and $N = mgcos\theta$. Thus a_{max} is found from

$$\begin{aligned} f_{max}r &= \frac{I_{cm}}{r}a_{max} \\ \mu mgcos\theta_{max} &= \frac{I_{cm}}{r}a_{max} \\ a_{max} &= \frac{r^2}{I_{cm}}\mu mgcos\theta_{max} \end{aligned}$$

Substituting a_{max} into the equation of exercise 3.1, gives:

$$\begin{aligned} \frac{g\ sin\theta_{max}}{1+\frac{I_{cm}}{mr^2}} &= \frac{r^2}{I_{cm}}\mu mgcos\theta_{max} \\ tan\theta_{max} &= \mu(1+\frac{mr^2}{I}) \end{aligned}$$

For a hoop, $tan\theta_{max} = 2\mu$, for a cylinder $tan\theta_{max} = 3$, and for a sphere, $tan\theta_{max} = 3.5\mu$. If θ is larger than these maximum angles, the object will not be able to roll without slipping. Try it out!

Exercise 3.3
A hoop starts from rest and rolls without slipping down an incline. The center of the hoop drops a distance h. What is the final translational speed of the hoop at the bottom of the incline (i.e. after the center of the hoop has dropped a distance h)?

We can use the work-energy theorem to solve this problem, since we are not interested in the time it takes the hoop to roll down the ramp. The net work done equals the change in kinetic energy. There are two forces acting: gravity and friction. The force of gravity down the ramp equals $mgsin\theta$, the force of friction up the ramp equals f. Therefore, the work done to change translational speed is $W_{translational} = (mgsin\theta - f)d$ where d is the distance traveled. Friction also does positive work in rotating the hoop. If the hoop rolls without slipping, then friction acts the same distance d in increasing the rotational kinetic energy. So the total work done in increasing the energy of motion is

$$W_{net} = (mgsin\theta - f)d + fd = mgdsin\theta = mgh$$

The net work equals the change in the kinetic energy. The initial K.E. is zero, so we have:

$$\begin{aligned} mgh &= \frac{m}{2}v^2 + \frac{I_{cm}}{2}\omega^2 \\ &= \frac{m}{2}v^2 + \frac{mr^2}{2}(\frac{v}{r})^2 \end{aligned}$$

For a hoop, $I_{cm} = mr^2$. If the hoop rolls without slipping then $\omega = v/r$. Carrying out the algebra gives

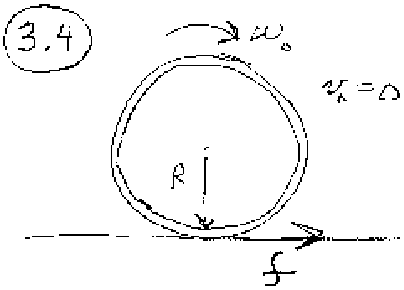
$$\begin{aligned} mgh &= \frac{m}{2}v^2 + \frac{mr^2}{2}(\frac{v}{r})^2 \\ mgh &= mv^2 \\ v &= \sqrt{gh} \end{aligned}$$

It is interesting to note that mgh is the initial potential energy of the system. This potential energy is converted into kinetic energy (translational plus rotational). That is, the mechanical energy of the system is conserved! The conservation of mechanical energy might not have been evident at first, since the force of friction is present. However, the negative work that friction does in decreasing the translational kinetic energy equals the positive work it does in increasing the rotational kinetic energy.

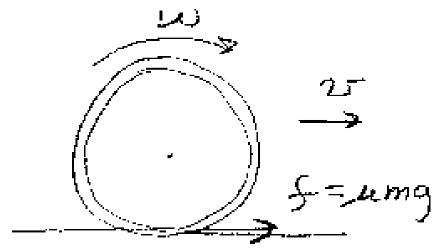
Exercise 3.4
Gilligan spins a hula hoop in front of himself and gives it a large initial angular velocity of ω_0. It lands and spins on the floor. Friction propels it forward and at the same time slows down the rotational speed. Eventually the hoops translational velocity matches its angular velocity and it rolls without slipping. What is the hoops final speed?

In this example, the hoops motion is one of translation and rotation. As discussed in lecture, the essential physics is: **The acceleration of the center of mass equals the net force; the angular acceleration about the center of mass equals the**

(3.4)



Initially



$$\tau = I\alpha$$

$$fR = (mR^2)\alpha$$

$$\mu mgR = mR^2\alpha$$

$$\alpha = \frac{\mu g}{R} \Rightarrow \boxed{\omega = \omega_0 - \frac{\mu g t}{R}}$$

$$f = ma$$

$$\mu mg = ma$$

$$a = \mu g$$

$$\boxed{v = \mu g t}$$

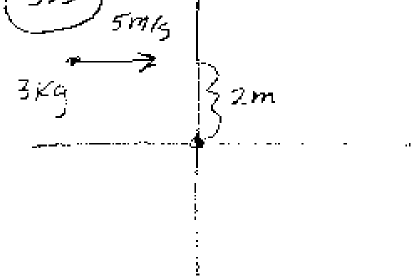
when $v = \omega R$ the hoop rolls without slipping

$$\mu g t = \left(\omega_0 - \frac{\mu g t}{R}\right)R$$

$$t = \frac{\omega_0 R}{2\mu g}$$

$$v_f = \mu g\left(\frac{\omega_0 R}{2\mu g}\right) = \boxed{\frac{\omega_0 R}{2}}$$

(3.5)



$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} = \vec{r} \times m\vec{v}$$

$$|\vec{L}| = (2m)(3kg)(5 m/s)$$

$$|\vec{L}| = 30 \; kg\frac{m^2}{s}$$

$\vec{L}$ is into the page

net torque about the center of mass.

There are three forces acting on the hoop: $m\vec{g}$, N (the normal force of the floor on the hoop, and f the frictional force. The only force parallel to the floor is the force of friction, f. Since the hoop is skiding on the floor, $f = \mu mg$, where μ is the coefficient of kinetic friction. Thus,

$$\begin{aligned} f &= ma \\ \mu mg &= ma \\ a &= \mu g \end{aligned}$$

As long as the hoop skids on the floor, the force of friction will accelerate the hoop. The velocity of the hoop as a function of time is

$$v = at = \mu gt$$

The angular acceleration about the center of mass equals the net torque about the center. $m\vec{g}$ acts at the center, and $\vec{N}$ points towards the center, so neither of these forces produces any torque about the center. Only friction exerts a torque about the center of mass, and is equal to fR where R is the radius of the hoop. Since the net torque equals $I_{cm}\alpha$,

$$fR = mR^2\alpha$$

where $I_{cm} = mR^2$ for the hoop. Since $f = \mu mg$, the angular acceleration about the center of mass is

$$\begin{aligned} \mu mgR &= mR^2\alpha \\ \alpha &= \frac{\mu g}{R} \end{aligned}$$

The frictional torque slows down the angular velocity of the hoop. The angular velocity ω as a function of time is

$$\omega = \omega_0 - \frac{\mu g}{R}t$$

As time goes on, ω will get smaller and v will increase until $v = \omega R$. When $v = \omega R$ then the hoop will roll without slipping. The time t when this happens is

$$\begin{aligned} \mu g t &= \omega_0 - \frac{\mu g t}{R} \\ t &= \frac{\omega_0 R}{2\mu g} \end{aligned}$$

Since the final velocity is $v_f = \mu g t$,

$$v_f = \mu g(\frac{\omega_0 R}{2\mu g}) = \frac{\omega_0 R}{2}$$

This is a rather amazing result. The final speed does not depend on the mass of the hoop, g, or the coefficient of kinetic friction μ.

Exercise 3.5
An small particle of mass 3 Kg is traveling in the x-y plane. Its velocity is $\vec{v} = 5\hat{i}$ m/s. When it is at the position $(-3, 2)$, what is the particle's angular momentum about the origin?

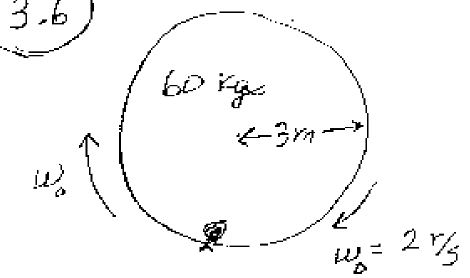
The angular momentum of a particle about an axis is $\vec{L} = \vec{r} \times \vec{p}$. The magnitude of the angular momentum is $|\vec{L}| = rpsin\theta = mv(rsin\theta)$. Note that $rsin\theta$ is the "moment arm", which is 2 m as shown in the figure. So the magnitude of the angluar momentum is

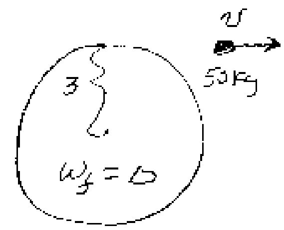
$$|\vec{L}| = 2(3)5 = 30\ Kg\ m^2/s$$

Since the "rotation" is clockwise, $\vec{L}$ points into the page.

Exercise 3.6
Lenny, mass 50 Kg, is spinning around on a merry-go-round in his local playground. The merry-go-round is a disk of mass 60 Kg and a radius of 3 meters. Lenny is standing on the edge, 3 meters from the center. The initial angular velocity of the merry-go-round is $\omega_0 = 2$ r/s. Suddenly, Lenny jumps off the merry-go-round so fast that it stops rotating. What is his speed with respect to the ground?

(3.6)





$m = 50$ Kg

<u>Before</u>

$L = I\omega_0 + 50(3)^2\omega_0$

$L = \frac{60}{2}(3)^2\omega_0 + 50(3)^2\omega_0$

$L = 720\,\omega_0 = 720(2) = 1440\ Kg\frac{m^2}{s}$

<u>after</u>

$L = 0 + 3(50)v$

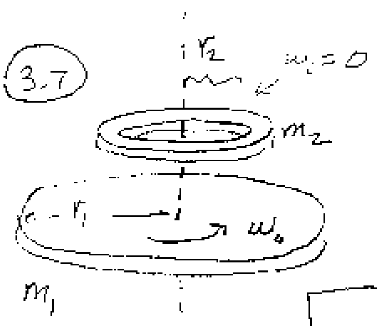
$L = 150v$

since the Forces are internal, angular momentum is conserved!

$150v = 1440$

$\boxed{v = 9.6\ m/s}$

(3.7)



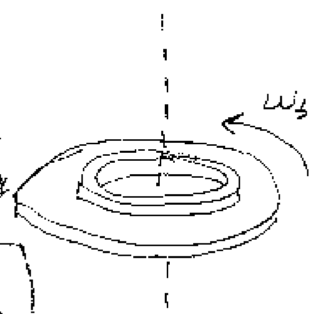
Before

$L_i = L_f$

$I_1\omega_0 = (I_1 + I_2)\omega_f$

$\frac{m_1}{2}r_1^2\omega_0 = \left(\frac{m_1}{2}r_1^2 + m_2 r_2^2\right)\omega_f$

$$\boxed{\omega_f = \left(\frac{m_1 r_1^2}{m_1 r_1^2 + 2m_2 r_2^2}\right)\omega_0}$$



after

Since all the forces are internal, so are the torques. Thus **angular momentum** is conserved. The angular momentum before Lenny jumps off is the same after he has jumped off. As a reference point, lets use the center of the merry-go-round The initial angular momentum is the sum of the angular momentum of the merry-go-round plus the angular momentum of Lenny about the center of the merry-go-round.

$$\begin{aligned} L_{initial} &= I_{disk}\omega_0 + mR^2\omega_0 \\ &= \frac{60}{2}3^2\omega_0 + (50)3^2\omega_0 \\ &= 270(2) + 450(2) \\ &= 1440\ Kg\frac{m^2}{s} \end{aligned}$$

The angular momentum after Lenny jumps off is also the sum of the angular momentum of the merry-go-round plus that of Lenny. Since the merry-go-round is stationary it has no angular momentum after Lenny jumps. However, Lenny will have angular momentum about the center of the merry-go-round equal to $\vec{r} \times m\vec{v}$. $\vec{r} \times m\vec{v}$ is equal to mv times the moment arm of 3 meters. So

$$L_{final} = mgr = 50(3)v = 150v$$

Since there are no external torques, angular momentum is the same before as after Lenny's jump:

$$\begin{aligned} L_{final} &= L_{initial} \\ 150v &= 1440 \\ v &= 9.6\ m/s \end{aligned}$$

Exercise 3.7
A flat disk of mass m_1 and radius r_1 is initially rotating with an angular velocity of ω_0. A ring of mass m_2 and radius r_2 is dropped on the spinning disk. After some sliding, the two objects rotate coaxially together with a final angular velocity of ω_f. Find an expression for ω_f in terms of the other parameters of the system.

Since the only force that the disk feels is due to the ring, and visa-versa, all the forces acting on the system are internal forces. Thus, the total angular momentum of the system is conserved. That means that the final angular momentum must equal the initial angular momentum. In each case, we have the situation in which a rigid object is rotating about a fixed axis. Thus, for each case, $L_{total} = I_{total}\omega$. Initially only the disk is rotating, so $I_{total} = I_{disk} = m_1 r_1^2/2$. After, the ring is also rotating, so $I_{total} = I_{disk} + I_{ring} = m_1 r_1^2/2 + m_2 r_2^2$. So the equations which result from the conservation of angular momentum are:

$$\begin{aligned} L_{initial} &= L_{final} \\ I_1\omega_0 &= (I_1 + I_2)\omega_f \\ \frac{m_1}{2} r_1^2 \omega_0 &= (\frac{m_1}{2} r_1^2 + m_2 r_2^2)\omega_f \\ \omega_f &= \frac{m_1 r_1^2}{m_1 r_1^2 + 2m_2 r_2^2}\omega_0 \end{aligned}$$

Exercise 3.8
Vickie has just received a yoyo for her birthday. She lets it unwind as it falls due to gravity. She is very curious as to what its acceleration is when it unwinds. She has asked us to help her calculate the yoyo's acceleration.

The falling yoyo is an object that is translating and rotating. The best way to analyze this kind of motion is to consider the center of mass. That is: **The acceleration of the center of mass equals the net force; the angular acceleration about the center of mass equals the net torque about the center of mass.**

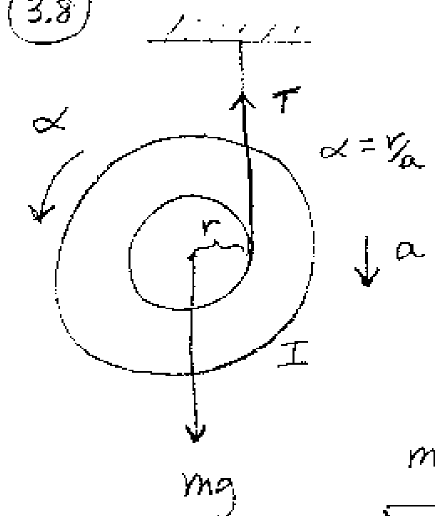
Let T be the tension in the string. There are two forces acting on the yoyo: mg downward, and T upward. Thus, the net force on the yoyo is $mg - T$ where down is the positive direction. From Newton's second law we must have:

$$mg - T = ma$$

where a is the acceleration of the center of mass (i.e. the center of the yoyo). The force of gravity, mg does not produce any torque about the center of mass, since it acts there. There is a torque about the center of mass due to T: rT. The "torque" equation yields:

$$rT = I_{cm}\alpha$$

(3.8)



$$mg - T = ma$$

$$Tr = I\alpha$$

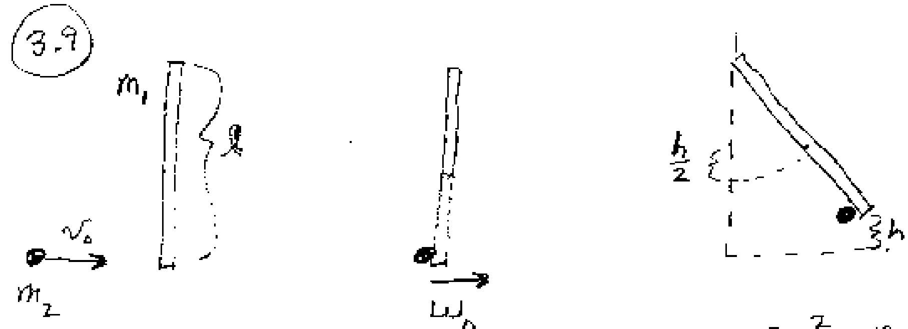
$$T = \frac{I}{r}\frac{a}{r}$$

$$T = \frac{I}{r^2} a$$

$$mg - \frac{I}{r^2} a = ma$$

$$a = \frac{g}{1 + \frac{I}{mr^2}}$$

(3.9)



$$L_i = m_2 v_0 \ell = L_f$$

$$m_2 v_0 \ell = I\omega_0$$

$$\omega_0 = \frac{m_2 v_0 \ell}{I}$$

$$\frac{I}{2}\omega_0^2 = m_2 g h + m_1 g h/2$$

$$\frac{I}{2}\left(\frac{m_2 v_0 \ell}{I}\right)^2 = \left(m_2 + \frac{m_1}{2}\right) g h$$

$$h = \frac{m_2^2 v_0^2 / g}{2\left(\frac{m_1}{3} + m_2\right)\left(m_2 + \frac{m_1}{2}\right)}$$

where α is the angular acceleration about the center of mass. However, a and α are not independent if the string doesn't strech. The yoyo unwinds as it falls such that $\alpha = a/r$ where r is the radius of the inner disk that the string is wound around. So we have

$$\begin{aligned} T &= \frac{I_{cm}}{r}\frac{a}{r} \\ &= \frac{I_{cm}}{r^2}a \end{aligned}$$

Substituting into the force equation gives:

$$\begin{aligned} mg - \frac{I_{cm}}{r^2}a &= ma \\ a &= \frac{g}{1 + I_{cm}/(mr^2)} \end{aligned}$$

Most yoyo's have a large I_{cm} and a small r^2, so their downward acceleration is much less than g.

Exercise 3.9
Consider the "ballistic stick" pendulum shown in the figure. A bullet of mass m_2 is initially traveling with a speed of v_0 towards a hanging stick that is at rest. The bullet penetrates into the bottom of the stick, then the stick (with bullet inside) swings upward. The stick has a length l and mass m_1. The maximum height of that the bullet (and bottom of stick) rise is h. Express h in terms of the masses of the system and v_0 and g.

We need to treat this problem in two parts. The first part is the collision of the bullet with the stick. During the collision, angular momentum about the pivot point is conserved. After the collision, the stick swings up to its maximum height h. During this part, mechanical energy is conserved.

First part: The angular momentum about the pivot point is the same before as after the collision. The angular momentum before the collision equals $m_2 v_0 l$. After the collision, the stick plus bullet swing about the pivot with an initial angular velocity of ω_0. The angular momentum about the pivot point is $I\omega_0$, where I is the rotational inertial about the pivot point: $I = I_{stick} + I_{mass} = m_1 l^2/3 + m_2 l^2$. Angular momentum conservation yields the following equation:

$$\begin{aligned} m_2 v_0 l &= I\omega_0 \\ \omega_0 &= \frac{m_2 v_0 l}{I} \end{aligned}$$

Second part: As the stick swings up, mechanical energy is conserved. The initial energy is rotational kinetic energy, $(I/2)\omega_0^2$. The final energy is all gravitation potential energy, which is $m_1 g$ times the distance that the center of mass of the stick is raised $(h/2)$ plus $m_2 g$ times the distance that the bullet is raised (h). Mechanical energy conservation yields the following equation:

$$\begin{aligned} \frac{I}{2}\omega_0^2 &= m_2 gh + m_1 g\frac{h}{2} \\ \frac{I}{2}\omega_0^2 &= (m_2 + \frac{m_1}{2})gh \end{aligned}$$

Note that if the bullet raises up a distance h, the center of the stick raises up a distance $h/2$. The above equations are essentially the work-energy theorem: the change in the kinetic energy equals the net work done by the gravitational force.

Substuting in to the angular momentum equation for ω_0 we have

$$\begin{aligned} \frac{I}{2}(\frac{m_2 v_0 l}{I})^2 &= (m_2 + \frac{m_1}{2})gh \\ \frac{m_2^2 v_0^2 l^2}{2(m_1 l^2/3 + m_2 l^2)} &= (m_2 + \frac{m_1}{2})gh \end{aligned}$$

Here we have used the expression for I about the pivot point, $I = m_1 l^2/3 + m_2 l^2$. Solving for h we have

$$h = \frac{m_2^2 v_0^2}{2g(m_1/3 + m_2)(m_2 + m_1/2)}$$

Note: the rotational inertial is the sum of the inertial of a rod that rotates about an end, $ml^2/3$, plus the mass of the bullet times l^2: $I = m_1 l^2/3 + m_2 l^2$.