Measuring c with an LC Circuit

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Maxwell's discovery of the relation between electricity, magnetism, and light was one of the most important ones in physics. With his added displacement current term, Maxwell showed that the equations of electricity and magnetism produced a radiation solution, electromagnetic (EM) radiation, that traveled with a speed of $c = 1/\sqrt{\varepsilon_0 \mu_0}$. The constant ε_0 is the permittivity of free space from electrostatics, and μ_0 is the permeability of free space from the magnetic interaction. There are a number of classroom experiments that directly measure the speed of light[1–3] and EM radiation[4] by dividing distance traveled by time. It is also possible to measure c using standing waves in a microwave oven.[5] These are excellent experiments, but do not demonstrate the relationship between c, ε_0, and μ_0. One can also measure the permittivity of substances in the classroom;[6] however, the quantity that is relevant for the speed of EM radiation is the product of $\varepsilon_0\,\mu_0$. The product $\varepsilon_0\,\mu_0$ is interesting since its units are *time*^{2}/*length*^{2}, and its value is the same for any choice of charge units. We present an experiment for the student laboratory to measure this product.

We determine the quantity $\varepsilon_0\,\mu_0$ by measuring the period of oscillation (or resonant frequency) of a capacitor-inductor (LC) circuit. The resonant angular frequency ω of an undamped LC circuit is given by $\omega = 1/\sqrt{LC}$, and the period of oscillation is $T = 2\pi\sqrt{LC}$, where L is the self inductance, which contains the constant μ_0, and C the capacitance of the circuit, which can be expressed using the constant ε_0.

Both the capacitor and inductor can be constructed by students. For the inductor, we made a solenoid by winding wire around a PVC pipe, and for the capacitor, we use a series of n identical flat sheet pairs of tin plates shown in Fig. 1. The capacitance for this configuration is $C = n\dfrac{\varepsilon_0 A}{d}$,

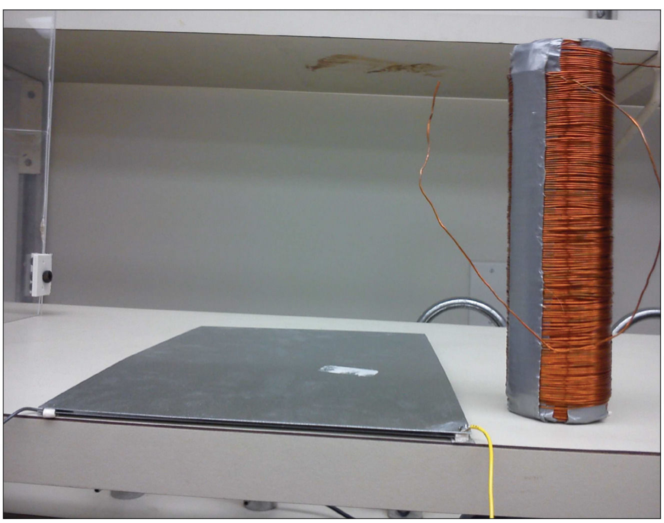

Fig. 1. A picture of our inductor and one capacitor plate-pair.

where A is the area of one pair of plates, n is the number of plates, and d is the plate separation. For the inductor,

$$L = \frac{\mu_0 N^2 A_L}{l},$$

where N is the number of turns, A_L is the cross-sectional area of the coil, and l is the length of the coil.[7] As an alternative to the homemade capacitor, one could purchase a variable capacitor that allows students to measure A and d, i.e., a ganged RF capacitor.

The relationship between the oscillation period T and the dimensions of the circuit elements is:

$$T = 2\pi\sqrt{LC} \tag{1}$$

$$T^2 = 4\pi^2 LC \tag{2}$$

$$= 4\pi^2\left(\frac{\mu_0 N^2 A_L}{l}\right)\left(\frac{\varepsilon_0 A}{d}n + C_0\right) \tag{3}$$

$$T^2 = \frac{4\pi^2 N^2 A_L A \mu_0 \varepsilon_0}{ld}n + T_0^2, \tag{4}$$

where C_0 is the capacitance of the oscilloscope, $A = 0.0667\ \mathrm{m}^2$ is the area of one plate, n is the number of capacitor plates connected, d is around 3 mm, and T_0 is the period of oscillation when only the oscilloscope is connected to the inductor (i.e., $n = 0$). In our case, $T_0 \approx 11.75\ \mu$s. If a graph of T^2 versus n yields a straight line, then the product $\varepsilon_0\,\mu_0$ can be determined from the value of the slope of the line.

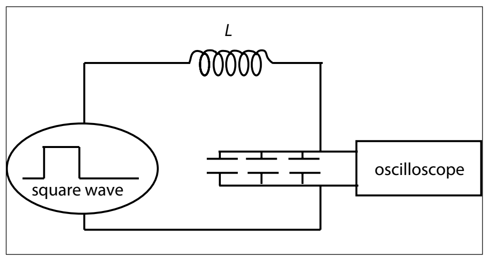

Fig. 2. Circuit diagram of the experiment. The voltage source on the left is a square wave. In the figure we have shown the case when *n* = 3.

A circuit diagram of the setup is shown in Fig. 2. The voltage source, coil, and oscilloscope are connected in series. The parallel plates are connected across the oscilloscope. The experiment consists of varying the capacitance by changing the number of plate-pairs and measuring the period of the oscillations of the circuit on the oscilloscope. The voltage source is a square wave, and a sample oscilloscope trace is shown in Fig. 3. In the figure, there are 16 cycles in 300 μs, resulting in a period T for the oscillations of approximately 18.75 μs.

The oscillations are initiated by using a square wave voltage. As an inexpensive source, we use a simple microcontroller, a PIC12F629, which costs under \$2. The microcontroller was programmed to repeatedly produce 3.2 V for 750 μs followed by 0 V for 750 μs. A frequency generator is

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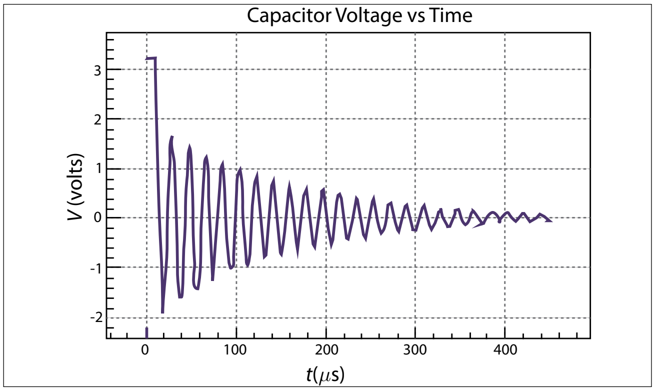

Fig. 3. A graph of the voltage on the osilloscope vs time. For this case there were six plates connected in parallel, n = 6. The period of one oscillation is approximately 18.75 μs.

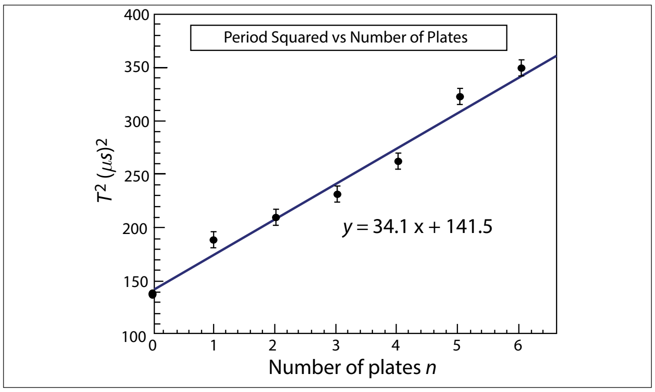

Fig. 4. A graph of T^2 vs the number of capacitor plates n. For the uncertainty in T, we have used ±0.25 μs.

another option for producing the square wave voltage. Alternatively, one could drive the LC circuit with a sinusoidal voltage and determine the resonant frequency. We chose the PIC microcontroller because of its low cost and the experience for the students of programming a microcontroller.

In order to achieve a period of the oscillations sufficiently large (resonant frequencies sufficiently small) for an accurate measurement on the oscilloscope, one needs L and C to be as large as reasonably possible. For the inductor, we wrapped 700 turns of wire around a PVC pipe with a diameter of 5 cm. The length of the coil of wire is 19 cm, resulting in a self-inductance of $L = (4\pi \times 10^{-7})700^2\pi\, 0.025^2/0.19 \approx 0.00636$ H, or approximately 6.36 mH.

The parallel plate pairs will be connected in parallel with the oscilloscope, so the capacitance of the oscilloscope will add to the circuit's total capacitance. We first measure the resonant frequency with just the oscilloscope present to obtain a data point with $n = 0$, and to estimate the capacitance of the oscilloscope. In this case, we obtain a resonant frequency of 85.1 kHz for $n = 0$. Thus, the capacitance of the oscilloscope can be estimated using $C = 1/(L\omega^2) \approx 5.5\times 10^{-10}$ F or around 550 pF. The parallel plate capacitors that we add need to have a capacitance comparable to this value to see their effect. We can get a rough idea how large the total plate area of the capacitors needs to be using $C = \varepsilon_0 A/d$. For a plate separation of 3 mm, we have $A \approx (5.5 \times 10^{-10})0.003/(8.85\times 10^{-12}) \approx 0.2$ m^2. For our experiment we use metal plates that are each 9 in x 12 in, giving an area of A ≈ 0.0667 m^2 for each plate. The two plates are set up with one on top of the other. Four small "spacers" are placed at the ends to keep the top plate from touching the bottom one. Six connected plate pairs in parallel will produce a total capacitance greater than that of the oscilloscope.

We display our data in Fig. 4. The square of the resonant period T^2 is plotted versus the number of parallel plates n connected across the oscilloscope. For example, the oscilloscope display of Fig. 3 has six parallel plates connected with a value of $T^2 \approx 18.75^2\ \mu s^2 \approx 352\ \mu s^2$ for $n = 6$. The linear plot is consistent with Eq. (4) and hence with Faraday's law, Coulomb's law, and Ampere's law, from which it was derived. Equating the slope of 34.1 $(\mu s)^2 = 3.41\times 10^{-11} s^2$ with

$$\frac{4\pi^2 N^2 A_L A\mu_0\varepsilon_0}{ld}$$

gives a value for $1/\sqrt{\varepsilon_0\mu_0}$ of

$$\frac{1}{\sqrt{\varepsilon_0\mu_0}} = 2\pi N\sqrt{\frac{A_L A}{(slope)ld}} \approx 3.6\times 10^8 \text{m/s}, \qquad (5)$$

where we have used d = 3 mm.

We estimate the uncertainty in this result to be around 20%. The largest contribution to the error stems from the estimation of d. Using a generous uncertainty of 1 mm for d, d = (3 ± 1) mm, gives a 33% error in its value. Since d is under the square root, the percent uncertainty is halved to 16%. In addition, the expressions for C and L are approximate. Edge effects in the capacitor and inductor have been neglected as well as the effect air has on the constants. With these errors, we estimate the result for $1/\sqrt{\varepsilon_0\mu_0}$ to be $(3.6\pm 0.7)\times 10^8$ m/s, a value within 20% of the accepted value 3×10^8 m/s.

We have neglected the small correction for ω due to damping. The correct expression for the angular frequency including damping is

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2},$$

where

$$\omega_0 \equiv 1\sqrt{LC}$$

and the amplitude of oscillation decreases as $e^{-\alpha t}$. We can estimate α from Fig. 3 by noting that it takes around five oscillations for the amplitude to decrease by a factor of $1/e$. Thus, $\alpha \approx 1/(5T) \approx \omega_d/(10\pi)$. Since $\omega_d \approx \omega_0$, we can estimate the correction to be

$$\omega_d \approx \omega_0\sqrt{1-(1/10\pi)^2} \approx 0.9995\omega_0.$$

It is important to point out that even though our value for $1/\sqrt{\varepsilon_0\mu_0}$ is close to the speed of light, we did not verify the existence of nor detect electromagnetic radiation. In fact, we did not directly measure the speed of any quantity. What then does this quantity with units of speed represent?

To answer this question, consider the origin of constants ε_0 and μ_0 in Eq. (4), which is derived from Faraday's law. The

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constant ε_0 that appears in the expression for the capacitance C was derived from Coulomb's law for the electrostatic force. The constant μ_0 appears in the expression for the self-inductance L, which came from the Biot-Savart law, or Ampere's law. Both of these constants arise from the force laws expressed with the electric and magnetic fields, and the same unit for charge. The quantity $1/\sqrt{\varepsilon_0\mu_0}$ derived from just the force laws will have units of speed and be the same value for any choice of charge units. Speed enters the picture due to the fact that the magnetic interaction depends on the speed (or current) of the source and the speed (or current) of the object.

This concept is most easily seen by considering an example often treated in introductory physics classes: the electric and magnetic forces between two infinitely long parallel line charges moving to the right with speed v and separated by a distance r. Let the charge per length of each rod be λ. Then the repulsive electrostatic force per length between the two charged moving rods is $F_e/l = E\lambda = \lambda^2/(2\pi\varepsilon_0 r)$. The attractive magnetic force between the two moving rods is $F_m/l = B\lambda v = \mu_0\lambda^2 v^2/(2\pi r)$, since the current is λv. Taking the ratio of the magnetic to the electric force for these two moving rods gives

$$\frac{F_m}{F_e} = v^2\varepsilon_0\mu_0. \tag{6}$$

For this example, $1/\sqrt{\varepsilon_0\mu_0}$ is the speed at which the magnetic force would equal the electrostatic force between the two moving rods. As with this example, in general, the speed $1/\sqrt{\varepsilon_0\mu_0}$ is a unique speed pertaining to the electric and magnetic forces that determines the relative importance of the two. It is this speed, from the force equations, that is being measured in the experiment. The force equations by themselves do not yield a radiative solution, which requires both Faraday's law and the addition of Maxwell's displacement current. Note that if the two rods are both moving to the right with $v > 1/\sqrt{\varepsilon_0\mu_0}$, then the two rods will attract each other, in contradiction with the reference frame in which they are stationary.

It is intriguing that the "ringing" that occurs in this electrical circuit allows one to measure a speed that does not depend on the length of the wires connecting the circuit elements. Rather, the speed depends on the dimensions of the volumes where the electric and magnetic forces are greatest and the fact that a changing magnetic field produces an electric field. That this speed is consistent with the speed of light reinforces for the student the connection of light with electricity and magnetism.

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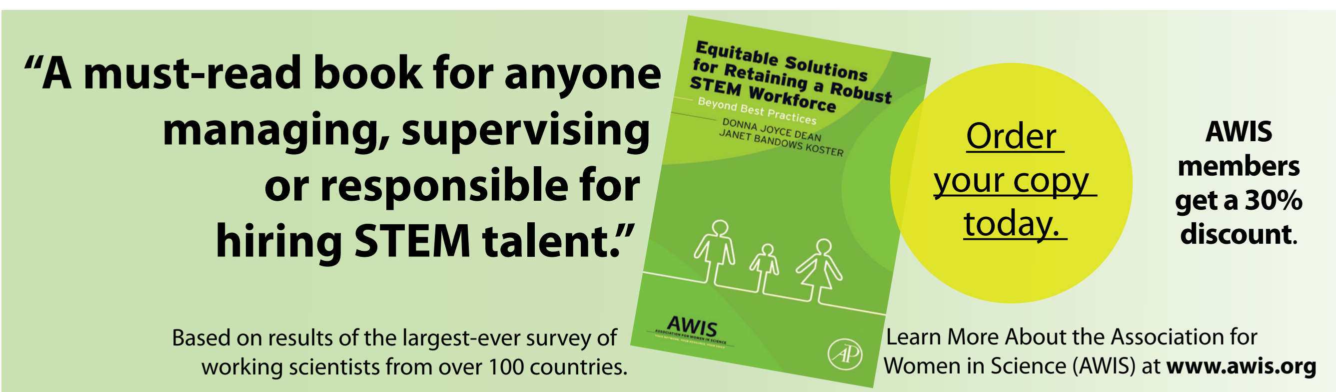

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