Sequential measurements and the commutator

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The commutator of two operators, defined as $[\hat{A},\hat{B}]\equiv\hat{A}\hat{B}-\hat{B}\hat{A}$, plays an important role in quantum mechanics. Two relationships derived in nearly all undergraduate texts are

$$\Delta A\Delta B \geqslant \tfrac{1}{2}|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|, \tag{1}$$

and the time evolution of the expectation value of an observable A

$$i\hbar\frac{d\langle\psi|\hat{A}|\psi\rangle}{dt} = \langle\psi|[\hat{A},\hat{H}]|\psi\rangle + i\hbar\langle\psi|\left(\frac{\partial\hat{A}}{\partial t}\right)|\psi\rangle, \tag{2}$$

where $|\psi\rangle$ is the state of the system, $\hat{H}$ is the Hamiltonian operator, and ΔA is the uncertainty in A. The operators $\hat{A}$ and $\hat{B}$ correspond to the observables A and B. The commutator also enters in the canonical quantization procedure and angular momentum algebra: $[\hat{J}_i,\hat{J}_j]=i\epsilon_{ijk}\hbar\hat{J}_k$. The $\hat{J}$ operators can be orbital, spin, or total angular momentum, and the commutation relations derive from the generators of SU(2).

In spite of the central role that commutation relations play in quantum mechanics, they are somewhat abstract and are usually not related to an experimental measurement. The purpose of this paper is to point out a special case for which the commutator is proportional to a measurable quantity: the probability of a change of state for sequential measurements. Because the commutator is the difference in the ordering of the operators of two observables, it might be suspected that successive measurements of the two observables are somehow related to the commutator. To understand such a possible relation, we need to determine the relevant quantities for successive measurements and the information contained in the commutator.[1,2] For our discussion, let the outcomes of observable A be a_i, with corresponding states $|\alpha_i\rangle$, and the outcomes of observable B be b_i, with corresponding states $|\beta_i\rangle$. Suppose the system starts in the pure state $|\alpha_1\rangle$. The probability P_1 that the system is found in a different state after B is measured is given by

$$P_1 = \sum_{i\neq 1,k}^{n} |\langle\alpha_i|\beta_k\rangle\langle\beta_k|\alpha_1\rangle|^2 = \sum_{i\neq 1,k}^{n} |C_{ik}|^2, \tag{3}$$

where $C_{ik}\equiv\langle\alpha_i|\beta_k\rangle\langle\beta_k|\alpha_1\rangle$, and n is the dimension of the state space. The off-diagonal matrix elements of the commutator $[\hat{A},\hat{B}]_{i1}$ in the $|\alpha_j\rangle$ basis are given by

$$\begin{aligned}\langle\alpha_i|\hat{A}\hat{B}-\hat{B}\hat{A}|\alpha_1\rangle &= (a_i-a_1)\langle\alpha_i|\hat{B}|\alpha_1\rangle\\ &= (a_i-a_1)\sum_k^n b_k\langle\alpha_i|\beta_k\rangle\langle\beta_k|\alpha_1\rangle\\ &= (a_i-a_1)\sum_k^n b_k C_{ik}.\end{aligned} \tag{4}$$

It is interesting that P_1 and $[\hat{A},\hat{B}]_{i1}$ depend on the exact same quantities, C_{ik}, where $2\leqslant i\leqslant n$ and $1\leqslant k\leqslant n$. However, the dependencies are fundamentally different. The commutator matrix elements are a coherent sum of the C_{ik} weighted by the eigenvalues b_k, whereas only the magnitudes of the C_{ik} enter in the sum for P_1. The unitarity of the $\langle\alpha_i|\beta_k\rangle$ transformation matrix places additional constraints on the C_{ik}: $\Sigma_k\langle\alpha_i|\beta_k\rangle\langle\beta_k|\alpha_1\rangle=\Sigma_k C_{ik}=0$ for all $i\neq 1$. Thus, there are $(n-1)$ commutator equations and $(n-1)$ equations from unitarity resulting in $2(n-1)$ equations containing the complex C_{ik}. Because, in general, there are $n(n-1)$ different C_{ik}, for values of $n>2$, the C_{ik} (and also P_1) cannot be uniquely determined from a knowledge of $[\hat{A},\hat{B}]_{i1}$.

For $n=2$ the situation is special, because there is only one independent off-diagonal element, C_{21}. The constraints imposed on a two-state system result in a direct connection between the commutator and the experimental quantity, P_1. In this case, $P_1=|\langle\alpha_2|\beta_1\rangle\langle\beta_1|\alpha_1\rangle|^2+|\langle\alpha_2|\beta_2\rangle\langle\beta_2|\alpha_1\rangle|^2$. From unitarity, $\langle\alpha_2|\beta_1\rangle\langle\beta_1|\alpha_1\rangle=-\langle\alpha_2|\beta_2\rangle\langle\beta_2|\alpha_1\rangle$, and hence

$$P_1 = 2|\langle\alpha_2|\beta_1\rangle\langle\beta_1|\alpha_1\rangle|^2. \tag{5}$$

For a two-state system, the commutator, being anti-Hermitian (if $\hat{A}$ and $\hat{B}$ are Hermitian), has only one independent element

$$\begin{aligned}&\langle\alpha_2|[\hat{A},\hat{B}]|\alpha_1\rangle\\ &\quad= (a_2-a_1)\langle\alpha_2|\hat{B}|\alpha_1\rangle\\ &\quad= (a_2-a_1)(\langle\alpha_2|\beta_1\rangle b_1\langle\beta_1|\alpha_1\rangle + \langle\alpha_2|\beta_2\rangle b_2\langle\beta_2|\alpha_1\rangle)\\ &\quad= (a_2-a_1)(b_1-b_2)\langle\alpha_2|\beta_1\rangle\langle\beta_1|\alpha_1\rangle.\end{aligned} \tag{6}$$

We combine Eqs. (5) and (6) and obtain

$$P_1 = \frac{2}{(a_1-a_2)^2(b_1-b_2)^2}|\langle\alpha_1|[\hat{A},\hat{B}]|\alpha_2\rangle|^2. \tag{7}$$

Note that $P_1=P_2$, because Eq. (7) is symmetric upon interchange of 1 and 2. Thus, the probability that the system changes its state due to the measurement B is proportional to the absolute square of the commutator matrix element $\langle\alpha_1|[\hat{A},\hat{B}]|\alpha_2\rangle$.

The calculation is symmetric in A and B, so we also have

$$|\langle\beta_1|[\hat{A},\hat{B}]|\beta_2\rangle|^2 = \frac{(a_1-a_2)^2(b_1-b_2)^2}{2}\sum_k |\langle\beta_1|\alpha_k\rangle \times\langle\alpha_k|\beta_2\rangle|^2. \tag{8}$$

If the commutator is zero, then a measurement of B does not change the outcome of a measurement of A. A larger value for the commutator results in a higher probability that the system will undergo a state change if B is measured. Because probabilities are less than or equal to one, we have an upper bound for the commutator,

$$|\langle\alpha_1|[\hat{A},\hat{B}]|\alpha_2\rangle| \leq \frac{|(a_1-a_2)||(b_1-b_2)|}{\sqrt{2}}, \tag{9}$$

which is similar to Eq. (1), the generalized uncertainty relation. Equation (1) is useful for a state that is not an eigenstate of A or B, whereas Eq. (9) holds for eigenstates of A (or B).

Equations (7)–(9) are applicable only for a two-state system. However, the two state system is a classic pedagogical example in many undergraduate texts.[1] The Stern-Gerlach experiment and magnetic resonance for a spin-1/2 particle are common examples. The connection between the commutator and sequential measurements is yet another interesting aspect of two-state systems.

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[1]David J. Griffiths, *Introduction to Quantum Mechanics*, 2nd ed. (Pearson Prentice-Hall, Upper Saddle River, NJ, 2005). The calculation presented here was motivated by Problem 3.27, p. 125, on sequential measurements.

[2]For a good summary of measurement theory in quantum mechanics, see Max Jammer, *The Philosophy of Quantum Mechanics* (John Wiley & Sons, New York, 1974).

Relativity, energy flow, and hidden momentum

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In 1995, Romer[1] wrote an inquiry regarding the momentum and energy flow of electromagnetic systems with particular reference to hidden momentum. The responses[2] to the inquiry, as well as earlier related work,[3] touched on conservation of energy, linear momentum, and angular momentum. However, none discuss what I believe is the crux of the matter, the conservation law associated with the invariance of classical electrodynamics under proper Lorentz transformation, which implies the constant velocity of the system center of energy.

In this connection, it was pointed out recently[4] that the application of external forces $\mathbf{F}_{\text{ext,i}}$ to the charged particles of classical electron theory leads to the relation

$$\sum (\mathbf{F}_{\text{ext,i}}\cdot\mathbf{v}_i)\mathbf{r}_i = \frac{d(U\vec{\mathcal{X}})}{dt} - c^2\mathbf{P}, \tag{1}$$

where U is the total (mechanical and electromagnetic) system energy, $\mathbf{P}$ is the system momentum, and $\vec{\mathcal{X}}$ is the displacement of the system center of energy. This relation is related to the continuous flow of energy in space and takes into account the fact that the spatial location where energy is introduced is relevant for relativistic systems. The relation also is relevant for understanding what is termed "hidden momentum."

Both Romer[1] and Griffiths[5] have given specific examples that are said to involve "hidden momentum." In these models, there is the flow of energy in the form of particle kinetic energy (Griffiths) or electromagnetic flux (Romer) which is maintained by forces that are external to the electromagnetic system whose energy and momentum is discussed. (The fact that the external force is supposed to arise from an electric field in Griffiths' example is irrelevant because the electromagnetic field is not included in energy or momentum calculations.) The energy U and center of energy $\vec{\mathcal{X}}$ of the electromagnetic system do not change in these models, $d(U\vec{\mathcal{X}})/dt=0$. Therefore, from Eq. (1), the momentum $\mathbf{P}$ arises from the energy flow provided by the external forces $\mathbf{F}_{\text{ext,i}}$. It is the putative return flow of energy associated with the external forces that is identified as the "hidden mechanical momentum" in Romer's example; it is the mechanical energy flow itself that is identified as "hidden mechanical momentum" in Griffiths' model.

The models of Romer and Griffiths are analogous to the problem introduced by Taylor and Wheeler,[6] where a conveyor belt moves energy from one end of a platform to the other. An energy flow must have an associated momentum flow, and also a change in the location of a center of energy. In the Taylor and Wheeler problem, the backflow of energy required to maintain the center of energy of the described system is provided by the backward motion of the platform on which the conveyer belt is mounted. The hidden momentum models of Romer and Griffiths do not make explicit the return energy-flow mechanisms, and therefore we cannot tell how the extended system will behave. "Hidden momentum" is an ambiguous term, which simply serves to excuse us from describing the full energy-flow relations that are crucial to the behavior of a relativistic system.

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[2] E. Comay, "Exposing 'hidden momentum,'" Am. J. Phys. **64**, 1028–1034 (1996); V. Hnizdo, "Hidden momentum and the electromagnetic mass of a charge and current carrying body," *ibid.* **65**, 55–65 (1997); "Hidden momentum of a relativistic fluid carrying current in an external electric field," *ibid.* **65**, 92–94 (1997); "Hidden mechanical momentum and the field momentum in stationary electromagnetic and gravitational systems," *ibid.* **65**, 515–518 (1997).

[3] See the references cited in Ref. 1, and also, V. Hnizdo, "Conservation of linear and angular momentum and the interaction of a moving charge with a magnetic dipole," Am. J. Phys. **60**, 242–246 (1992); "Comment on 'Torque and force on a magnetic dipole,' by L. Vaidman, [Am. J. Phys. **58**, 978–983 (1990)]," *ibid.* **60**, 279–280 (1992).

[4] T. H. Boyer, "Illustrations of the relativistic conservation law for the center of energy," Am. J. Phys. (to be published).

[5] D. J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice Hall, Upper Saddle River, NJ, 1999), pp. 520–521.

[6] E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (W. H. Freeman, San Francisco 1966), pp. 147–148.