A simple setup to observe attractors in phase space

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(Received 15 June 1992; accepted 25 October 1992)

In a recent article[1] we discussed a simple experiment for the student laboratory in which period doubling, chaos, and the transient properties of nonlinear systems can be investigated. We present here an improvement in the data taking which allows a two-dimensional Poincaré plot of the motion to be accurately measured. With this improvement, detailed plots of strange attractors for chaotic behavior and attractors for periodic motion are obtainable. For stable systems, plots of attractors are produced by recording the return to steady state after a perturbation is applied. Figure 1 is a photo of the device, which consists of a magnetic needle free to rotate about its axis. The angular velocity of the needle is measured for a fixed angle by means of photogates consisting of an infra-red emitter and detector. Two photogates are needed to determine both the magnitude and direction of the needle's angular velocity. The compass is placed on top of a solenoid and between a set of Helmholtz coils. A direct current flows through the solenoid, while the needle is driven by a magnetic field produced by a sinusoidal current in the Helmholtz coils. This note reports on a simple way to use an analog to digital (A to D) card to accurately record a Poincaré section of the needle's motion.

The magnetic field from the Helmholtz coils forces the compass needle to spin about its axis, and data is recorded when one side of the needle blocks one of the photogates. When the blocking is completed, the angular velocity, ω, of the needle and the phase, ϕ, of the driving force are measured in succession. A plot is then made on the computer screen of ω versus ϕ, enabling the student to see the Poincaré section develop in "real" time (see Figs. 2–4).

The data are collected via both the digital to digital (D to D) and analog to digital ports of the card. The D to D port measures the time the photogate is blocked, which is inversely proportional to the needle's angular velocity. The A to D port measures the voltage of the function generator which is driving the compass. The blocking time measurement is facilitated by extending one side of the compass needle with aluminum foil (see Fig. 1). Gate number 1 is set so that only the aluminum foil will block it, thus it will count only one side of the needle. Gate number 2, which is about 170 deg in angle from gate 1, is set so that either side of the needle will block it. The digital port is read in a loop which keeps repeating as long as gate 1 is blocked. The angular velocity is easily determined, since it is inversely proportional to the repetitions of the loop. As soon as gate 1 becomes unblocked, the digital port is read again to determine if gate 2 is blocked. Since gate 2 is not directly opposite of gate 1, whether it is blocked or unblocked will depend on the direction of the needle's rotation. Thus both the magnitude and direction of the needle's angular velocity can be determined very accurately. For optimum results, the relative orientation of gates 1 and 2 will depend on the thickness of the needle.

The phase of the driving force, ϕ, which is the other dimension in the Poincaré plot, is determined by measuring the voltage across the Helmholtz coils just after the photogate measurements. The A to D port is connected across the Helmholtz coils and is read two times in succession, giving the values V_1 and V_2. The magnitude of the phase is $\sin^{-1}(V_1/V_{\max})$, where $V_{\max}$ is the amplitude of the sinusoidal voltage across the Helmholtz coils. The quadrant of the phase angle is found by comparing V_1 and V_2. If V_1 is positive, then $0°\leqslant\phi\leqslant 90°$ if $V_1 \leqslant V_2$ or $90°\leqslant\phi\leqslant 180°$ if $V_1 \geqslant V_2$. Likewise, if V_1 is negative, then $-90°\leqslant\phi\leqslant 0°$ if $V_1\leqslant V_2$ or $-180°\leqslant\phi\leqslant -90°$ if $V_1 \geqslant V_2$. This method is simple and works well, except when ϕ is very close to $\pm 90°$. At these angles, $d(\sin\phi)/d\phi$ is zero and resolution is limited. This can be seen in the strange attractor of Fig. 4 in which 2000 channels span the peak-to-peak voltage. Since $\sin^{-1}$ (0.999) is about 87.5°, the resolution at $\pm 90°$ is 2.5°. The poor resolution at these two angles, however, does not take away from the beauty of the attractor. The problem could be eliminated by reading the voltage from a sawtooth function triggered by the sine wave.

Examples of Poincaré plots for different values of $V_{\max}$ and solenoid currents are shown in Figs. 2–4. The driving frequency was 1.5 Hz for all the plots. These pictures, which form on the computer screen as the needle spins, display the angular velocity versus ϕ for a fixed photogate angle. The first few points are numbered according to the order in which they appear on the screen. The ω–ϕ plane is a different Poincaré section than is usually plotted for a driven nonlinear oscillator. The conventional approach is to plot the needle's position, θ, versus angular velocity, ω, for a fixed phase of the driving force.[2] We choose the former coordinates because it is difficult to obtain accurate measurements of the angular position of the needle using a simple photogate setup. We were able to reduce the noise level of the angular velocity to less than 0.3%, and using a 4000 channel A to D card gave us good resolution on the

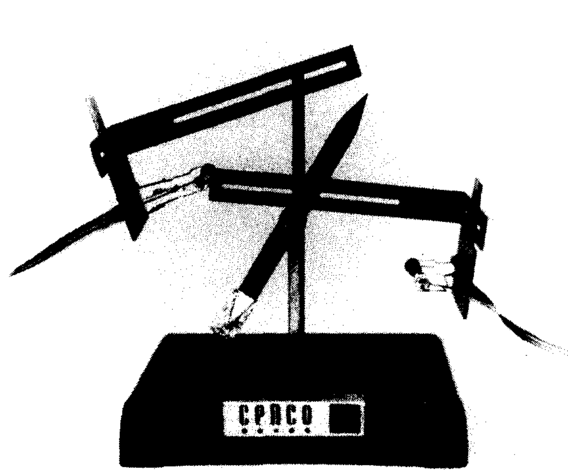

Fig. 1. Picture of the apparatus. The photogate on the right is referred to as gate number 1 in the text, and the one on the left as gate number 2.

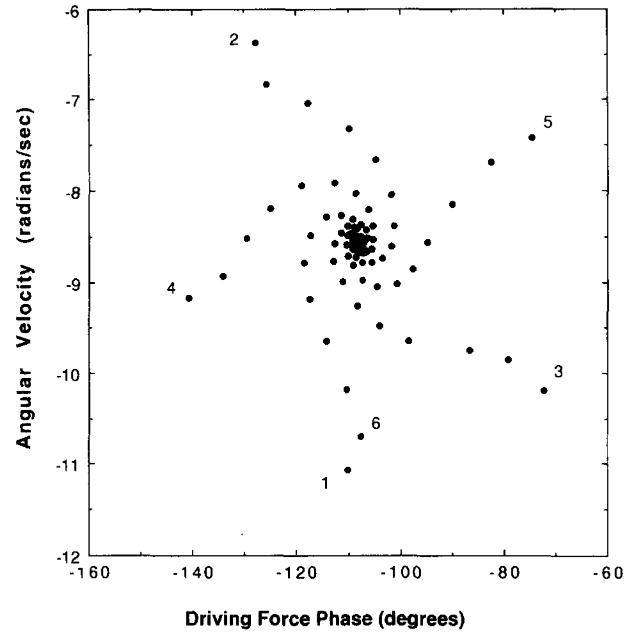

Fig. 2. Experimental data of a period 1 attractor. The system was perturbed, and the return to steady state was recorded. The transient winding angle[1] is about 144°.

phase of the driving force. With this accuracy some fine detail in the strange attractor for the needle undergoing chaotic motion was observed (see Fig. 4).

There are some features of this setup which make it particularly suited for the classroom. The interfacing of the A to D card to the apparatus and writing the software can be done by students and make an excellent project.[3] Being a mechanical system, the setup offers a good visual picture of a chaotic system. The students are always excited and amazed to see the strange attractor (Fig. 4) develop from the wildly spinning compass needle in front of them. Since the system is lightly damped the transient motion lasts a long time, and the parameters of the transient motion, for example the Jacobian of the nonlinear map, can easily be measured.[1] This is clearly demonstrated in the attractors of Figs. 2 and 3, for which the system was perturbed and its return to steady state recorded. The perturbation was applied by bringing a strong horseshoe magnet quickly near and quickly away from the vicinity of the spinning magnet. In Fig. 2, the transient winding angle is about 144° and the system experienced a bifurcation when the driving force was increased. In Fig. 3, the transient motion for a system undergoing period two motion is shown. Here, the transient winding angle is about 35° which indicates that the system just underwent a bifurcation.[1,4] In this figure, the points follow the order $1\rightarrow 1'\rightarrow 2\rightarrow 2'\rightarrow 3\rightarrow 3'\rightarrow$etc.. Finally, the best advantage is that the apparatus is inexpensive, and in our case was constructed from already existing equipment.

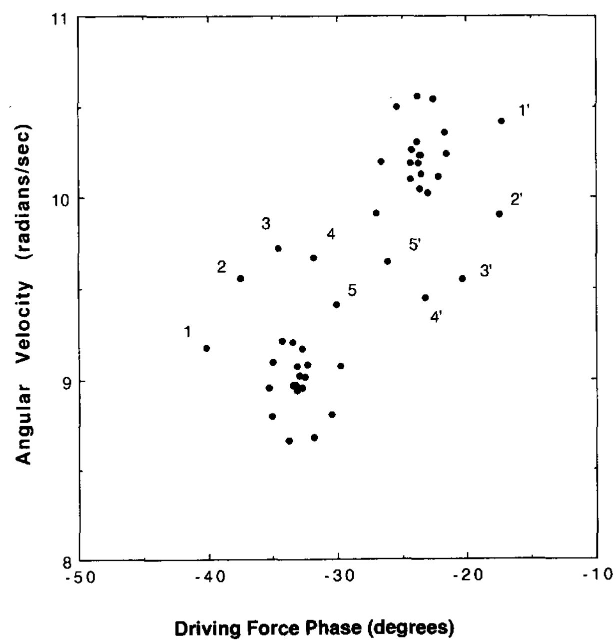

Fig. 3. Experimental data of a period 2 attractor. As in Fig. 2, the system was perturbed, and the return to steady state was recorded. The transient winding angle is about 35°.

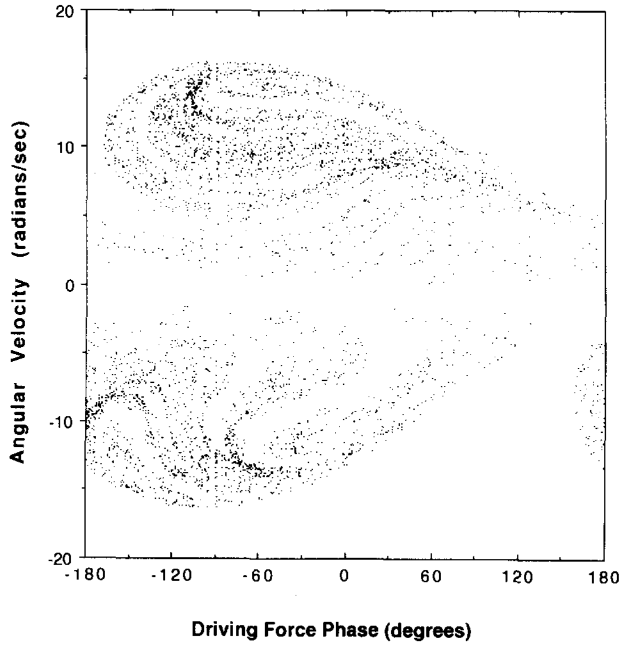

Fig. 4. Experimental data of a strange attractor. The data, which consist of 4000 points, were collected during a student laboratory class.

We would like to thank Eric Trout, Mark Southard, Chris Anderson, and Derek Holcomb for allowing us to use their laboratory class data. The project was partly funded by National Science Foundation ILI Grant USE-9152011 and a Cal Poly Research Scholarship Creativity Award.

[1]A. Ojah, S. Moon, B. Hoeling, and P. B. Siegel, "Measurements on the transient motion of a simple nonlinear system," Am. J. Phys. **59**, 614–619 (1991).

[2]See for example G. L. Baker and J. P. Gollub, *Chaotic Dynamics: An Introduction* (Cambridge University, New York, 1990).

[3]C. A. Kocher, "A laboratory course in computer interfacing and instrumentation," Am. J. Phys. **60**, 246–251 (1992).

[4]P. B. Siegel, "Measuring transient properties in dissipative systems," Phys. Rev. A **45**, 4192–4193 (1992).