Measuring transient properties in dissipative systems

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The transient motion of dissipative systems depends on the eigenvalues of the Jacobian of the map representing the system. In the high bifurcation limit of period-doubling systems, these eigenvalues approach universal curves in the period-doubling interval. We calculate these curves for two-dimensional iterative maps and compare with experimental measurements of the eigenvalues.

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Theoretical work on the properties of two-dimensional dissipative systems has been discussed in the literature [1–7]. It has been shown [1] that an effective period-doubling constant, δ_{eff}, can be defined that depends only on the effective dissipation B_{eff}. This dependence was shown to be universal in the high bifurcation limit. Effective constants and scaling functions which arise in the period-doubling route to chaos have also been calculated [2–7]. In this paper, we extend these theoretical ideas and calculate a set of universal functions: the eigenvalues of the Jacobian for two-dimensional iterative maps. These functions have particular importance since they are directly measurable quantities.

The eigenvalues of the Jacobian were calculated following the method of Refs. [1–5] by using the standard form of the Hénon map:

$$x_{n+1}=2Cx_n+2Cx_n^2-y_n,\quad y_{n+1}=Bx_n\ , \tag{1}$$

where C is the control parameter. The constant B is the Jacobian of the noninterated map. For a fixed value of B, the map undergoes period doubling as C is increased. The various period-doubling constants can be determined, and are found to depend in the high bifurcation limit only on the Jacobian of the iterated map, $B_{\text{eff}}=B^{2^k}$, where 2^k is the number of iterations [1]. The main properties of the bifurcation scheme can be understood by examining the eigenvalues of the Jacobian of the iterated map at the stable point. There are two eigenvalues for a two-dimensional map which we label as $\lambda_+=ae^{+i\Omega}$ and λ_-. The locus of these eigenvalues in the complex plane as the system goes through period doubling is discussed in Refs. [2,8,9]. It is shown that $\lambda_+\lambda_-=B_{\text{eff}}$, and that if the eigenvalues are complex, $\lambda_-=\lambda_+^*$.

The parameters a and Ω were determined numerically using Eq. (1) by solving for the eigenvalues of the Jacobian at the stable point. Calculations were performed for up to 2^6 iterations, which were sufficient for convergence. These numerical results are shown graphically in Fig. 1 for different values of δ_{eff} as defined in Eq. (3). The x axis corresponds to the fraction of the period-doubling interval of the control parameter:

$$x=\frac{C-c_k}{c_{k+1}-c_k}\ , \tag{2}$$

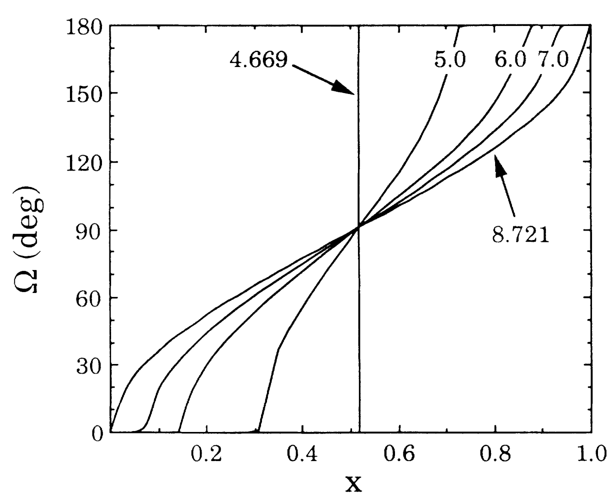

FIG. 1. Numerical results for the variation of the angle Ω through the period-doubling interval for different δ_{eff}. The curves are labeled with their respective values of δ_{eff}: 4.669, 5.0, 6.0, 7.0, and 8.721. The variable x is defined in Eq. (2).

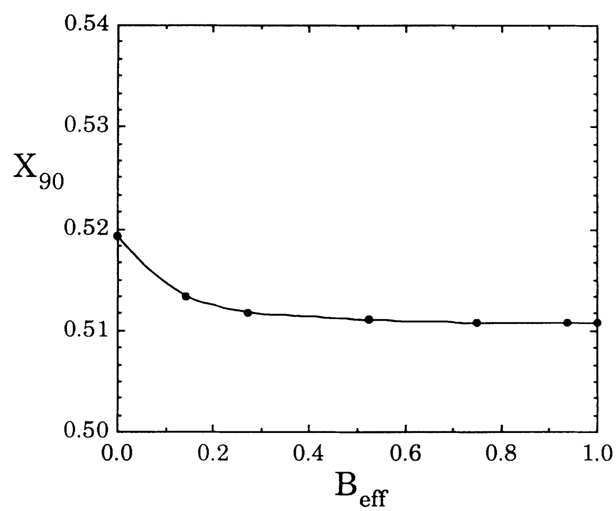

FIG. 2. A graph of X_{90}, defined in the text, vs B_{eff}. The points are numerical calculations for 64 iterations of the Hénon map. A smooth curve is drawn through the points. $X_{90}(0)=0.5193\ldots$ and $X_{90}(1)=0.5108\ldots$.

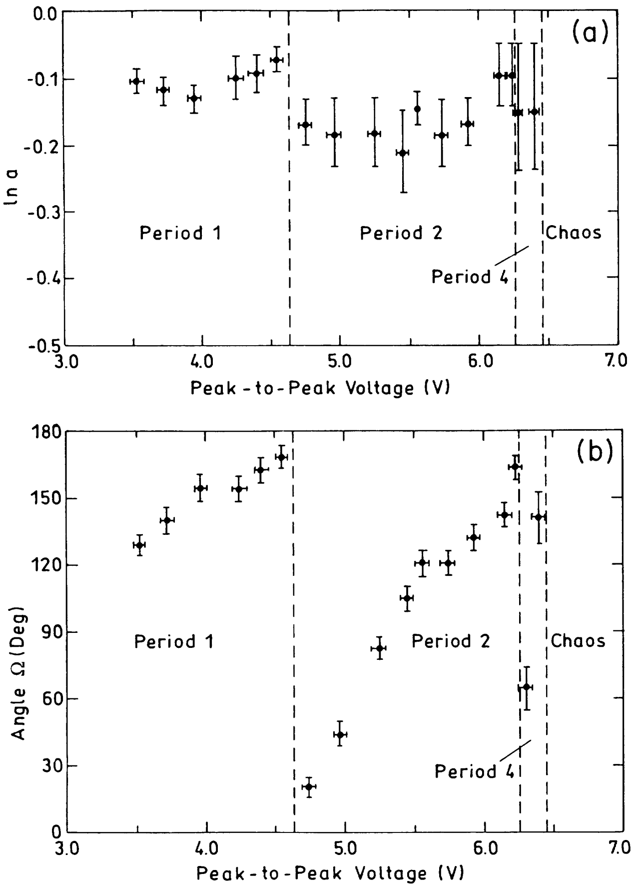

FIG. 3. A graph of the experimental measurements of the parameters (a) $\ln a$ and (b) Ω as a function of the peak-to-peak voltage across the Helmholtz coils.

where c_k is the value of C at which the map bifurcates into a 2^k cycle. The y axis is the value of $\Omega_{\delta_{\text{eff}}}$. The angle Ω changes from 0° to 180° as the parameter C varies through the period-doubling interval (i.e.,$0 \le x \le 1$). We have plotted $\Omega_{\delta_{\text{eff}}}(x)$ for values of δ_{eff} equal to 4.669 ($B=0$), 5.0 ($B_{\text{eff}}=0.046$), 6.0 ($B_{\text{eff}}=0.18$), 7.0 ($B_{\text{eff}}=0.32$), and 8.721 ($B=1$), as labeled in the figure. These functions $\Omega_{\delta_{\text{eff}}}(x)$ are universal in the high bifurcation limit for all two-dimensional maps undergoing period doubling. In our numerical calculations we find that the function $\Omega(x)$ determined from 64 iterations of $B=0.99$ and 32 iterations of $B=0.98$ were equal to within 1% for all values of x.

We have labeled the curves in Fig. 1 in terms of the value of δ_{eff} instead of B_{eff} for an easier comparison with the experimental data. The quantity δ_{eff} is defined in the period-2^k interval as

$$\delta_{\text{eff}}=\frac{\Delta C(\text{period}-2^k\text{interval})}{\Delta C(\text{period}-2^{k+1}\text{interval})}=\frac{c_{k+1}-c_k}{c_{k+2}-c_{k+1}} . \tag{3}$$

This is a slightly different definition for δ_{eff} than that given in Ref. [1], which uses the maximally stable points instead of the bifurcation points. We use the above definition, since in experiments one usually measures the bifurcation points rather than the maximally stable points. Note that the value of B_{eff} in the period-2^k interval is the square root of B_{eff} in the period-2^{k+1} interval.

It is interesting to note that in Fig. 1, all the curves have roughly the same value of x when Ω equals 90°, i.e., Re $(\lambda_\pm)$ equals zero. We define this value of x as $X_{90}(B_{\text{eff}})$, and plot it in Fig. 2 as a function of B_{eff}. It is seen that this parameter varies little in the crossover from dissipative to conservative systems.

It is important to point out that the eigenvalues $\lambda_\pm$ are directly measurable quantities. We measured $\ln a$ and Ω for a simple nonlinear lightly dissipative system as it underwent period doubling. The experiment is described in Ref. [10], and we show in Fig. 3 the results for comparison with the curves of Fig. 1. The curves of Fig. 1 are only applicable in the high bifurcation limit. However, even in the period-2 region the measured value of Ω and $\ln a$ are consistent with these curves. Comparing the data for the period-2 region in Fig. 3(b) to the curves of Fig. 1, we see that δ_{eff} is about 8. Determining δ_{eff} from the data also gives $\delta_{\text{eff}}\approx(6.26-4.62)/(6.46-6.26)\approx 8.0$. In addition, in Fig. 3(a) for the period-2 region we have $B_{\text{eff}}=a^2=e^{2\ln a}\approx e^{2(-0.2)}=0.67$, from the data. This effective dissipation value is also compatible with a value for δ_{eff} of around 8. Thus, by simply measuring Ω in only one periodic region one can determine δ_{eff}, quantify the dissipation, and examine the properties of a nonlinear system.

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