Air Friction Analysis and Drag Coefficients for Cylinders

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Abstract

We present an inexpensive experiment for the student laboratory to measure how the force of air friction on a cylinder depends on its velocity. The cylinder is attached as the bob of a long pendulum, with a laser gate to measure the time and its speed at the bottom of its arc as it swings back and forth. We show that the force of air friction as a function of the velocity can be expressed with only a linear and a quadratic term in the given velocity range. By varying the cylinder's height and diameter, students can gain insight into the nature of the drag coefficient C_d, an important quantity in the analysis of air friction.

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1 Introduction

The force of air friction is of great importance in many areas of physics and engineering, from the design of cars, airplanes, and rockets to the damping of oscillatory motions. The subject is therefore an essential part of every physics and engineering curriculum. Damping due to air friction of a physical pendulum, in particular, has been investigated both experimentally [1] for the student lab and theoretically [2] for advanced student projects. Other examples include a sphere falling in a fluid [3]. The investigations focus on the question whether the drag force depends linearly or quadratically on the speed of the object, given the range of the Reynolds number. For an analysis of these problems, it is typically necessary to find solutions to differential equations, a mathematical tool beginning students are often not familiar with. In this paper, we use a method of analyzing pendulum data in order to find the velocity dependence of the air frictional force, without explicitly solving a differential equation. This also allows us to compute the drag coefficient of the pendulum bob, a cylinder.

In introductory physics and engineering classes, the force of air friction is introduced with an ad hoc formula, stipulating that it is either proportional to the velocity or to the square of the velocity, depending on the velocity range in question. The drag coefficient C_d plays an important role in this equation. It is said to be a constant term plus a term inversely proportional to the velocity depending on the Reynolds number [4].

Our goal in this paper is to provide a motivation for these formulas based on experimental data, without the need to solve a differential equation. By analyzing measurements of a pendulum with a cylindrical bob, where air friction is essentially the only dissipating force, we are able to gain insight into air friction in a way that usually requires much more expensive and elaborate equipment such a wind tunnel [5] .

Our apparatus is a $2.2\,m$ long pendulum consisting of a V-shaped string with a cylinder moving perpendicular to its axis attached as the bob. As the cylinder swings back and forth gradually slowing down, a laser gate accurately measures the period of oscillation and the speed of the cylinder at the bottom of the swing by the blocking time of the gate.

The first task for the students is to determine the functional dependence of the frictional force on the velocity, $f(v)$. When the cylinder is stationary, $f(0) = 0$. We assume that the magnitude of the air frictional force $f(v)$ on the cylinder can be can be expressed in a Taylor series expansion, i.e. as the sum of polynomials of its speed v:

$$f(v) = k_1 v + k_2 v^2 + k_3 v^3 + \cdots \quad (1)$$

with the direction of the force opposite to that of its velocity. Our goal is to compute the factors k_1, k_2, k_3, $\cdots$, i.e the coefficients of the polynomial expansion of the air frictional force, thereby determining how many terms in the polynomial are necessary for an accurate fit to the data.

The quantity a^* that the students use in our experiment has units of acceleration and is defined as follows:

$$a^* \equiv \frac{2(v_1 - v_3)}{T} \tag{2}$$

Here v_1 is the initial speed of the cylinder at its lowest point, v_3 the speed at the end of a complete cycle, and T is the period of the pendulum (i.e. the time difference of the recordings of v_1 and v_3). For each swing, a^* is directly computed from the recorded data. We point out that the quantity a^* is not the instantaneous acceleration at any position of the pendulum; a^* is actually twice the acceleration due to air drag averaged over one cycle, i.e. an effective quantity related to the energy loss during one oscillation.

In the appendix, it is shown that

$$ma^* \approx k_1 v_2 + \frac{8}{3\pi} k_2 v_2^2 + \frac{3}{4} k_3 v_2^3 + \cdots \tag{3}$$

where the factors k_i are the same as in Eq.(1), m is the mass of the cylinder, and v_2 is the speed of the pendulum when it blocks the gate half way through a complete cycle. Dividing both sides of equation (3) by the mass m and fitting the a^* vs v_2 data with a polynomial allows us to determine the coefficients k_i in Eq.(1) for the polynomial expansion of the drag force.

In the appendices we show that this approximation is accurate to better than one percent for starting angles less than 30°. We point out that examining the dependence of a^* on v_2 allows us to determine the air frictional force directly from the data, without solving the differential equation $F = ma$ to model the motion.

The students are then able to determine how $f(v)$ depends on the cylinder's height and diameter, and thus make the connection between the force of air friction and the drag coefficient.

2 The Experimental Setup

For the experimental setup, we chose a cylinder moving perpendicular to its axis for a number of reasons. First, since the sides of the cylinder are parallel, the laser gate is blocked cleanly yielding consistent measurements. Second, the frictional force on the top and bottom are roughly the same for cylinders with the same diameter and different heights h. This property allows one to extrapolate the force linearly down to $h = 0$ to obtain the air frictional force due to the pendulum strings and attachment hooks. In addition, the laminar flow around the cylinder can be analyzed in two dimensions, a problem that can be covered in the classroom.

Fig. 1 shows a photo of the experimental set-up, as well as schematic of the pendulum.

A light string is suspended from two points at the rafters of the laboratory ceiling, and the cylindrical bob is attached to it. The string thus forms a V, and the pendulum, with a length of 2.2 meters, swings in a well defined plane. As the cylinder swings back and forth through a laser gate, the times that the gate is blocked, t_b, and unblocked, t_{ub}, are recorded by an Arduino board. Similar set-ups with laser gate and Arduino board have been described elsewhere [6, 7]. After each blocking and un-blocking of the gate, the two times are sent to a personal computer. The speed of the cylinder with diameter d, $(t_b - t_{ub})/d_e$, is computed, where the effective blocking diameter d_e is determined using the calibration method presented in [8]. In addition, the time $(t_b + t_{ub})/2$ at which the cylinder crosses the gate is recorded each time. After the pendulum is swinging so low as to barely block the gate, the data collection is stopped. The computer then calculates the quantities a^* and T, and writes a file of a^* versus v_2 for analysis.

To check the validity of Eq.(3) and determine the number of polynomials required for a good fit, the students graph and fit a^* versus v_2. The fit is done with a second order polynomial, $a^* = c_1 v_2 + c_2 v_2^2$, and for comparison with three terms, $a^* = c_1 v_2 + c_2 v_2^2 + c_3 v_2^3$. Here, the c_i are the coefficients of the polynomial fit of a^* versus v_2. Their relationship with the k_i of the drag force polynomial expansion can be seen by comparison of Eqs.(1) and (2): $c_1 = \frac{k_1}{m}$, $c_2 = \frac{1}{m}\frac{8}{3\pi}k_2$, and $c_3 = \frac{1}{m}\frac{3}{4}k_3$.

Results from a typical experiment are shown in Fig. 2. In our lab we use cylinders made from wooden rods since they come to rest after a shorter time. In the appendix, we show a similar graph using a metal cylinder.

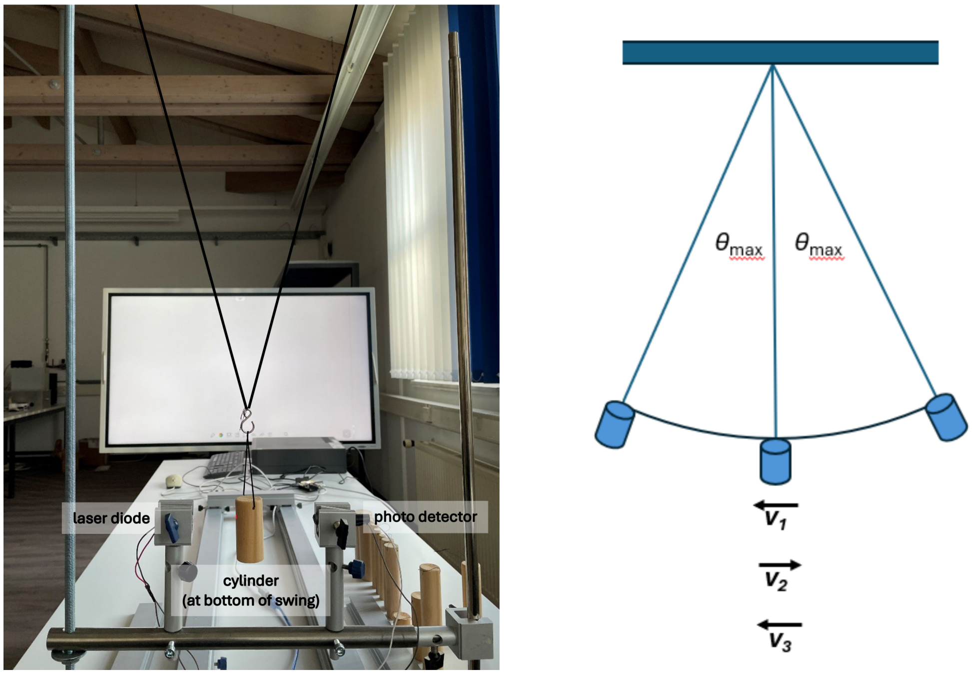

Figure 1: Left: The experimental setup. The string is attached at two points on the rafters forming a V. The wooden cylinder is connected by a hook to the V and swings into and out of the plane of the paper. Right: A schematic of the pendulum, as viewed perpendicular to the photo. The velocities are measured at the bottom of the swing. The time intervall between v_1 and v_3 is the period T, v_2 is the bottom speed in the middle of a cycle. The maximum angle of displacement θ_{max} decreases with each half swing.

The students can see that the two curves are practically identical, a visual demonstration that the cubic term is unnecessary: The air frictional force as a function of speed for the cylinder is well described by a linear plus quadratic term. For an advanced class, the reduced χ^2 can be calculated for a more quantitative proof as discussed in the appendix.

3 Data Analysis

After establishing the velocity dependence of the air friction, the students determine the dependence of k_1 and k_2 on the dimensions of the cylinder.

The first set of experiments is to examine how the coefficients k_1 and k_2

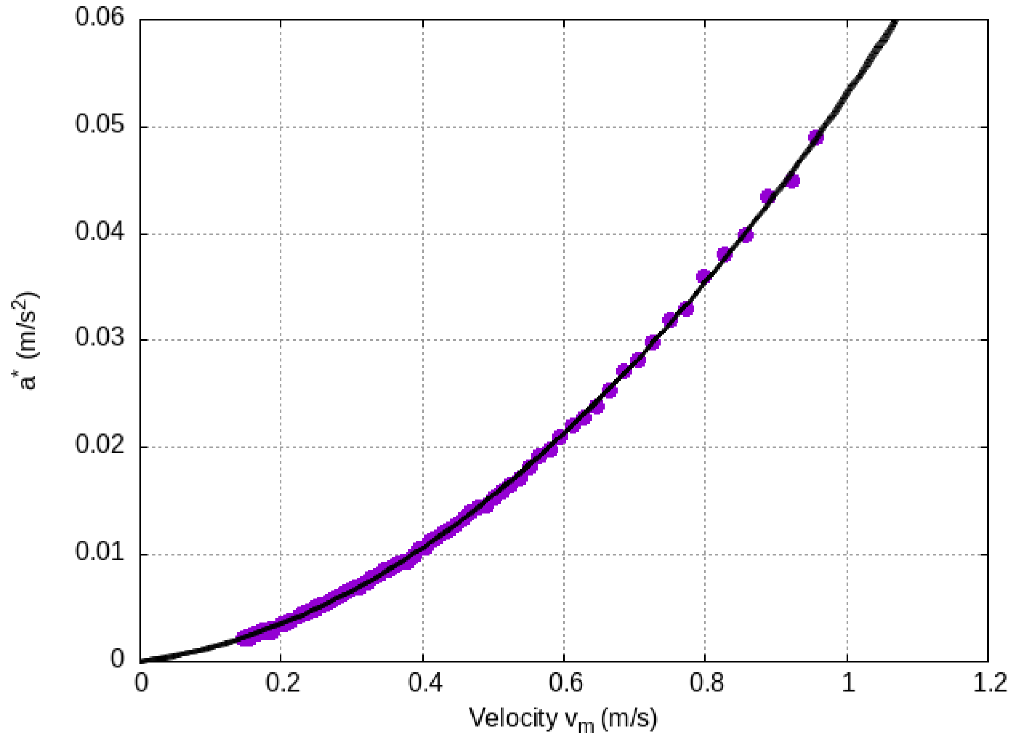

Figure 2: A graph of a^* versus speed v_2. The line for the linear plus quadratic fit and the line that includes the cubic term overlap completely, a visual demonstration that the cubic term is not necessary.

in Eq.(1) depend on the height h of the cylinder for the same diameter d. In our lab, we used cylinders of diameter 2.8 cm and heights of 5.8, 9.0, 12.0, and 14.8 cm. To obtain k_1 and k_2, a^* has to be multiplied by the mass m of the cylinder. From the fit parameters c_1 and c_2, we determine $k_1 = mc_1$ and $k_2 = m\frac{3\pi}{8}c_2$, according to Eq.(3). The students then plot k_1 and k_2 as a function of h. Typical plots are shown in Figs. 3a and 3b, demonstrating that both k_1 and k_2 are proportional to the cylinder height h.

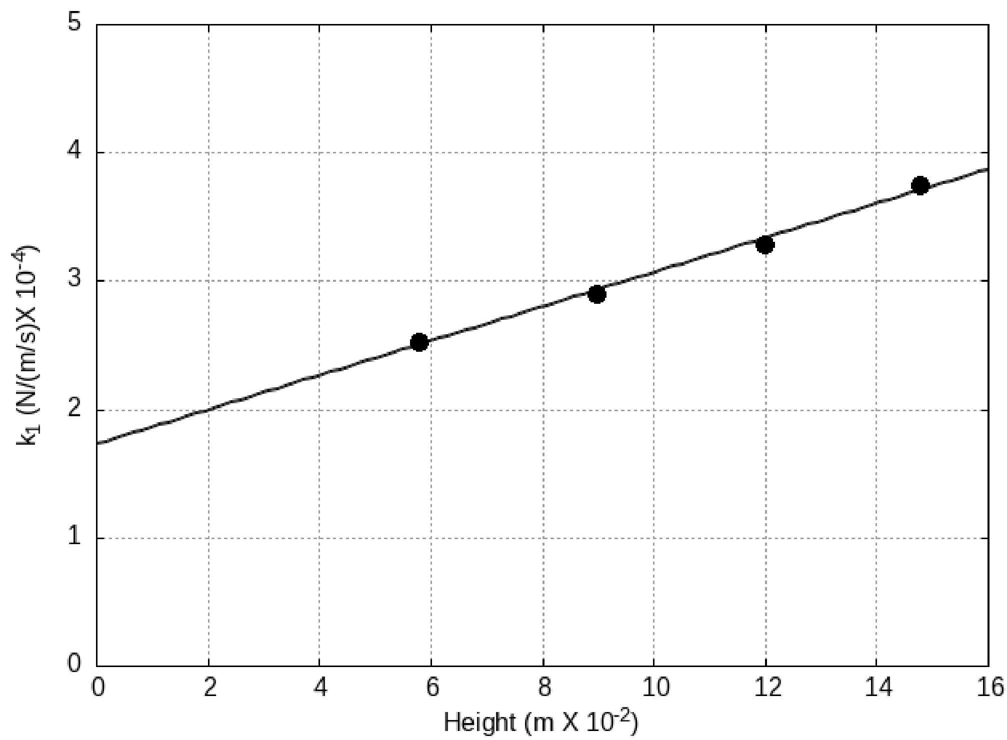

Figure 3a: A graph of k_1 versus the height h for a common diameter $d = 2.8$ cm. The solid line is the linear fit: $k_1 = (1.73 + 0.134 \cdot h) \times 10^{-4} \frac{Ns}{m}$, where h is in cm.

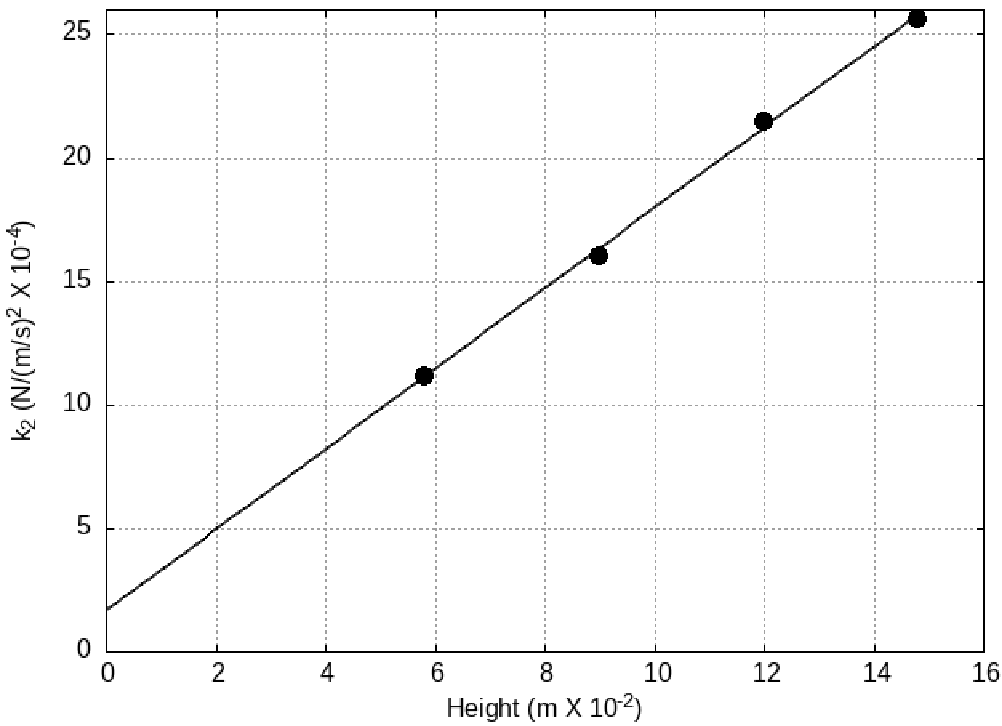

Figure 3b: A graph of k_2 versus the height h for a common diameter $d = 2.8$ cm. The solid line is the linear fit: $k_2 = (0.17 + 1.63 \cdot h) \times 10^{-4} \frac{Ns^2}{m^2}$, where h is in cm.

The linear relationship in h is expected, since a doubling of the height is the same as two cylinders one on top of the other. The vertical intercepts are the values of k_1 and k_2 for the string and attachment hook.

The next set of experiments is to examine how k_1 and k_2 depend on the diameter d of the cylinder for a fixed height. The students used cylinders with a height of 12 cm and diameters of 1.4, 1.8, 2.2, 2.5, 3.4 and 4.0 cm. It is not obvious that the air friction should be proportional to d for both k_1 and k_2. However, the plots of k_1 and k_2 as a function of d for $h = 12$ cm shown in Figs. 4a and 4b confirm that this is indeed the case: Both k_1 and k_2 also depend linearly on the cylinder's diameter d.

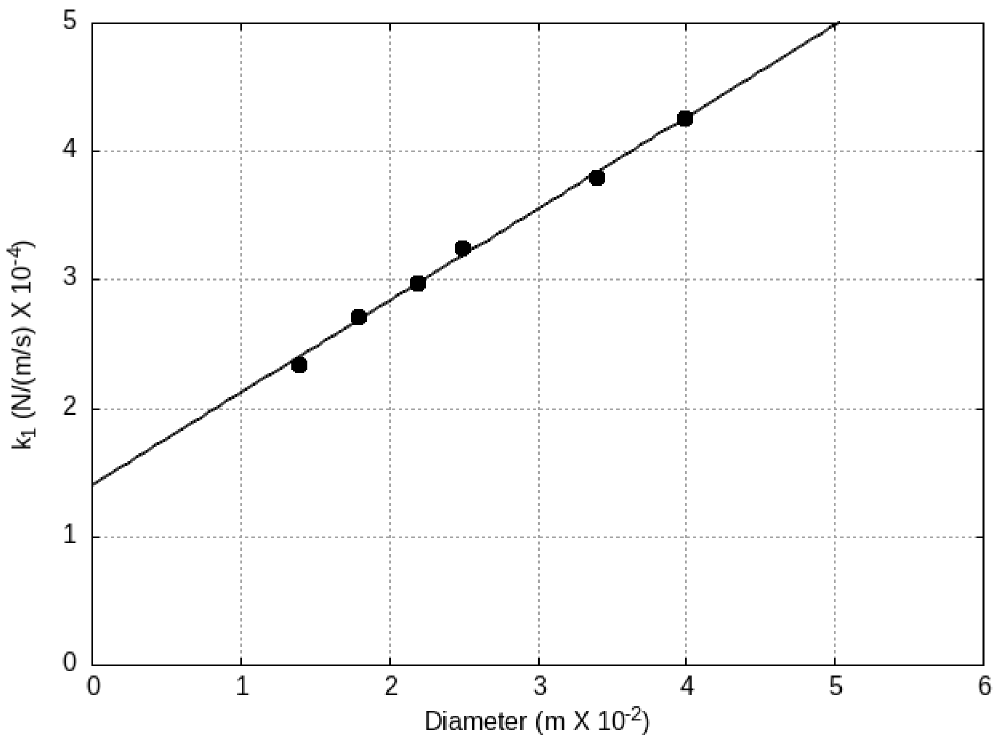

Figure 4a: A graph of k_1 versus the diameter d for a common height $h = 12$ cm. The solid line is the linear fit: $k_1 = (1.40 + 0.72 \cdot d) \times 10^{-4} \frac{Ns}{m}$, where d is in cm.

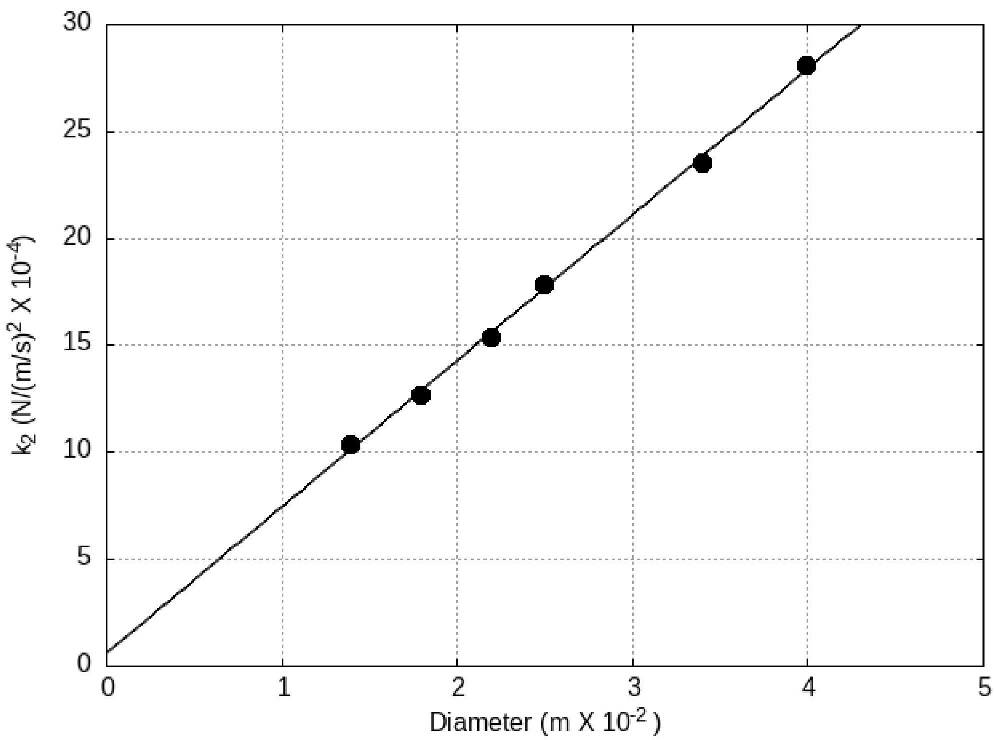

Figure 4b: A graph of k_2 versus the diameter d for a common height $h = 12$ cm. The solid line is the linear fit: $k_2 = (0.6 + 6.83 \cdot d) \times 10^{-4} \frac{Ns^2}{m^2}$, where d is in cm.

For the cylinder alone (without hook and string), both the v and v^2 dependence of the frictional force are proportional to both h and d. Hence, each piece is proportional to the product hd or the frontal area A of the cylinder. This is a very nice result, since one can factor out the area A from both terms and plot and fit k_1 and k_2 as a function of the area A. The result of these fits are shown in Figs. 5a and 5b, where A has units of cm^2.

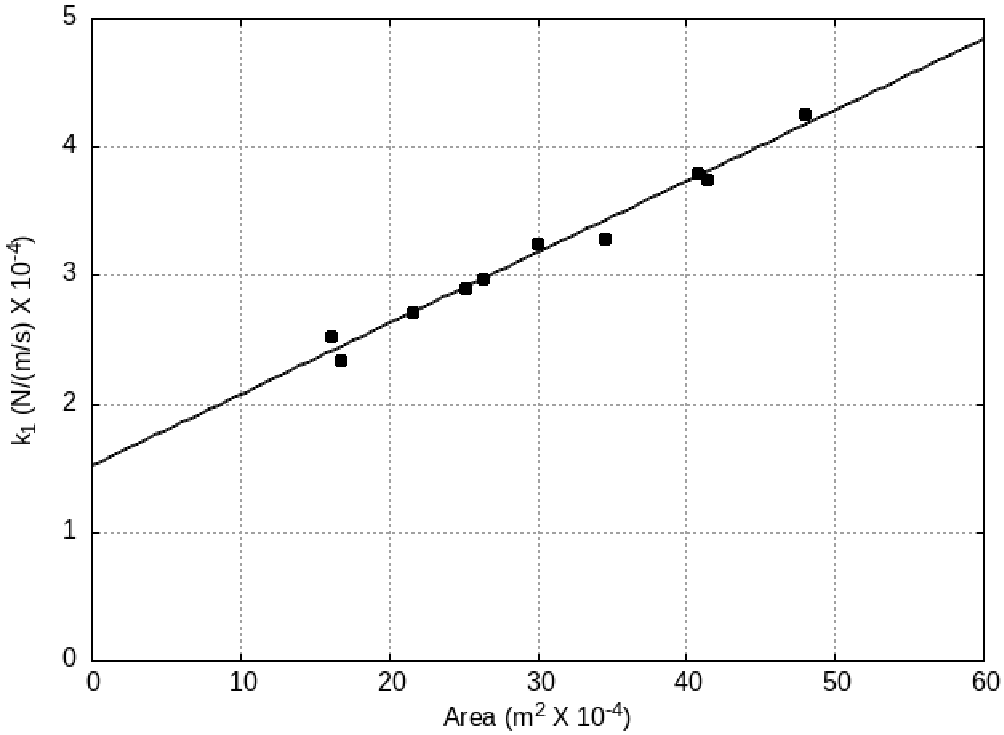

Figure 5a: A graph of k_1 versus the Area A. The solid line is the linear fit: $k_1 = (1.52 + 0.055A) \times 10^{-4} \frac{Ns}{m}$, where A is in cm^2.

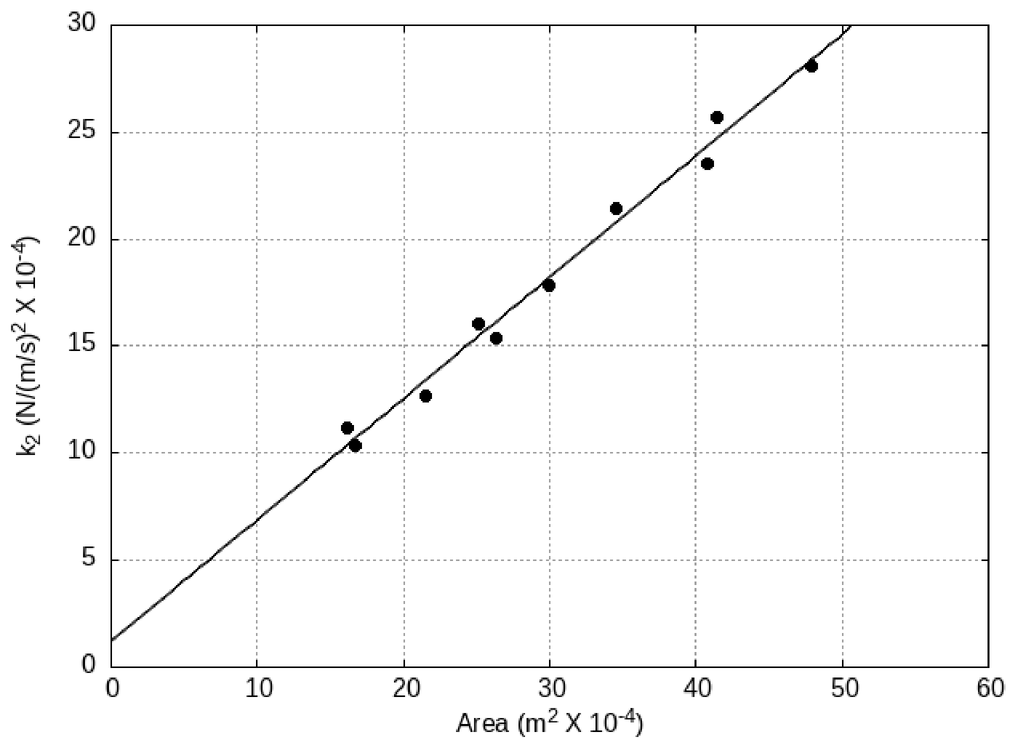

Figure 5b: A graph of k_2 versus the Area A. The solid line is the linear fit: $k_2 = (1.2 + 0.567A) \times 10^{-4} \frac{Ns^2}{m^2}$, where A is in cm^2.

Table 1: Dependence of k_1 and k_2 on the cylinders' diameter d (in cm), height h (in cm) and area A (in cm^2).

	$k_1 \times 10^{-4} \frac{Ns}{m}$	$k_2 \times 10^{-4} \frac{Ns^2}{m^2}$
Figs. 3: $d = 2.8\ cm$	$1.73(\pm 0.04) + (0.134 \pm 0.004) \cdot h$	$1.7(\pm 0.4) + 1.63(\pm 0.04) \cdot h$
Figs. 4: $h = 12\ cm$	$1.40(\pm 0.06) + (0.72 \pm 0.02) \cdot d$	$0.6(\pm 0.3) + 6.83(\pm 0.12) \cdot d$
Figs. 5:	$1.52(\pm 0.09) + (0.055 \pm 0.003) \cdot A$	$1.2(\pm 0.7) + 0.567(\pm 0.023) \cdot A$

In Table(1), we summarize the values for the three fits. The intercepts agree fairly well with each other. We estimate the frictional force due to the hook and string using the values from the graphs of k_1 and k_2 versus the area A in Fig. 5a and Fig. 5b:

$$f_{string+hook} \approx (1.52v\frac{1}{m/s} + 1.2v^2\frac{1}{(m/s)^2}) \times 10^{-4}N \tag{4}$$

For the cylinders, we estimate the frictional force coefficients k_1 and k_2 to be the values of the slopes of Figs. 5a and 5b:

$$f_{cylinder} \approx (0.055v\frac{1}{m/s} + 0.567v^2\frac{1}{(m/s)^2})A\ \frac{N}{m^2}, \tag{5}$$

with A in units of cm^2. This two-term model has been considered previously in [3, 8]. It is common practice to factor out v^2, and express the force of air friction as

$$f_{cylinder} \approx (\frac{0.055(m/s)}{v} + 0.567)v^2 A\ \frac{Ns^2}{m^4} \tag{6}$$

Thus, the students can establish that the air frictional force on a cylinder is proportional to the frontal area A and to the (unitless) quantity in parentheses. Note that the combination $N\frac{s^2}{m^4}$ has units of mass density. Since the air friction is also expected to be proportional to the density of air, ρ, this quantity, with a factor of 1/2, is multiplied with the product v^2A. Therefore, the two terms in parentheses have to be divided by $\rho/2$, where we have taken the density of air in our lab room to be $\rho_{air} = 1.225\ \frac{kg}{m^3}$.

$$f_{cylinder} \approx (\frac{0.090(m/s)}{v} + 0.926)\frac{\rho}{2}Av^2$$

The quantity in parentheses is the velocity dependent unitless drag coefficient C_d. We obtain the equation for the air frictional force of a cylinder in the well-known form in terms of the drag coefficient C_d [9]:

$$f_{cylinder} \approx C_d\frac{\rho}{2}Av^2$$

with our experimental value of the velocity-dependent drag coefficient C_d for a cylinder moving perpendicular to its axis:

$$C_d \approx \frac{0.090(m/s)}{v} + 0.926 \quad (7)$$

Our measured drag coefficients C_d for speeds less than 1.2 $\frac{m}{s}$ agree well with the literature values, up to Reynolds numbers of $100,000$ [10], even though our data only go up to $Re \approx 6,000$.

4 Conclusion

We have described an inexpensive experiment for undergraduate physics and engineering laboratory classes to measure the air frictional force on a cylinder that is moving with its axis perpendicular to its velocity. By examining the quantity a^*, the students can determine the air frictional force directly from the data without having to model the motion with the differential equation, $F = ma$. The experiment shows that the force is well described by two terms, one linear and one quadratic in the cylinder's speed. The students can measure how the air frictional force depends on the cylinders diameter d and height h. The result is that for each term, linear and quadratic, the force is proportional to the product dh or the frontal area A. These findings motivate the ansatz for describing the force with the help of a coefficient, called the drag coefficient C_d, which is the sum of a constant term and a term inversely proportional to v. The force of air friction then takes on the well-known form $F = C_d\frac{\rho}{2}Av^2$, which shows its proportionality to the frontal area A.

5 Acknowledgments

We would like to thank Florian Federmann and Martin Stemberger for their support with the equipment of the experiment.

Appendix I: Derivation of the quantity a^*

From Newton's law of motion, the equation for the motion of the swinging cylinder is

$$I\frac{d^2\theta}{dt^2} = -lmg\ sin(\theta) + lf(v) \tag{8}$$

where $f(v)$ is the force of air friction on the cylinder, which is always in the direction opposite to the motion. We model the magnitude of the force of air friction to be $|f(v)| = k_1|v| + k_2v^2 + k_3|v^3| + \cdots$. The students' data consist of the speed and time of the cylinder when it crosses the laser gate at the bottom of its swing. One could numerically solve Newton's equation of motion and vary the parameters k_1, k_2, k_3, etc. to best fit the data. However, we show below that if the swing angle θ is small, one can directly obtain the air friction parameters k_i by using the work-kinetic energy theorem.

As the laser gate is located at the bottom of the cylinder's swing, there will be three blockings of the gate for a complete cycle of the pendulum. We label v_1 as the speed at the start of a cycle. The speed in the middle of the cycle when the pendulum swings back is labeled as v_2. The final speed at the end of the cycle is v_3 (which is the same as v_1 for the next cycle). The work-kinetic energy theorem, which the students learn in their introductory physics classes, states that the change in the kinetic energy of an object equals the net work done on it:

$$\frac{m}{2}(v_3^2 - v_1^2) = \int_1^3 \vec{f} \cdot d\vec{r} \tag{9}$$

where the scalar product $\vec{f} \cdot d\vec{r}$ is negative. Using $d\vec{r} = \frac{d\vec{r}}{dt}dt = \vec{v}dt$, we have

$$\frac{m}{2}(v_3^2 - v_1^2) = \int_{t_1}^{t_3} \vec{f} \cdot \vec{v}\ dt \tag{10}$$

and

$$\frac{m}{2}(v_1^2 - v_3^2) = \int_{t_1}^{t_3} (k_1 v^2 + k_2|v^3| + k_3 v^4 \cdots) dt \tag{11}$$

Since the air frictional force on the cylinder is small, the motion of the pendulum bob for a single swing can be described as an harmonic oscillation. For a simple (undamped) pendulum, the restoring force is given by $\vec{F}(\theta) = -mg\ sin(\theta)\ \hat{\theta}$, where θ is the angle the pendulum makes with the vertical, and $\hat{\theta}$ is the unit vector in the direction of $+\theta$. For angles $\theta < 15°$, $sin(\theta) \approx \theta$ to better than 1%. Thus, the restoring force is to a very good approximation proportional to the pendulum's displacement, resulting in simple harmonic motion for the swinging cylinder. The velocity as a function of time *without air friction* can therefore be written as:

$$\vec{v}(t) \approx v_{max}\ cos(\frac{2\pi}{T}t)\hat{\theta} \tag{12}$$

where T is the period of the pendulum, and v_{max} is the speed at the bottom of the swing. We apply the work-kinetic energy theorem to a complete cycle of the swing, starting when the pendulum is at the bottom of its swing with the velocity v_1.

Substituting these quantities into the equation above we have the approximate result:

$$\frac{m}{2}(v_1^2 - v_3^2) \approx \int_0^T [k_1 v_{max}^2 cos^2(\frac{2\pi}{T}t) + k_2 v_{max}^3 |cos^3(\frac{2\pi}{T}t)| + k_3 v_{max}^4 cos^4(\frac{2\pi}{T}t) + \cdots] dt$$

$$\frac{m}{2}(v_1^2 - v_3^2) \approx v_{max} \int_0^T [k_1 v_{max} cos^2(\frac{2\pi}{T}t) + k_2 v_{max}^2 |cos^3(\frac{2\pi}{T}t)|) dt + k_3 v_{max}^3 cos^4(\frac{2\pi}{T}t) + \cdots] dt$$

However, v_{max} decreases slowly every half-swing due to air friction: $v_1 > v_2 > v_3$. There are different possibilities for choosing v_{max} in the above equation. A simple choice is to let v_{max} equal v_2, since this value occurs in both the first and second half swings. With this choice, the equation is

$$\frac{m}{2}\frac{(v_1^2 - v_3^2)}{v_2} \approx \int_0^T [k_1 v_2 cos^2(\frac{2\pi}{T}t) + k_2 v_2^2 |cos^3(\frac{2\pi}{T}t)|) dt + k_3 v_2^3 cos^4(\frac{2\pi}{T}t) + \cdots] dt \tag{13}$$

The right hand side can be further simplified using $v_1^2 - v_3^2 = (v_1 - v_3)(v_1 + v_3)$ and, to a very good approximation, $v_2 \approx \frac{v_1+v_3}{2}$:

$$m(v_1 - v_3) \approx \int_0^T [k_1 v_2 cos^2(\frac{2\pi}{T}t) + k_2 v_2^2 |cos^3(\frac{2\pi}{T}t)|)dt + \; k_3 v_2^3 cos^4(\frac{2\pi}{T}t) + \cdots]dt \tag{14}$$

Carrying out the integrals, we obtain

$$m\frac{2(v_1 - v_3)}{T} \;=\; k_1 v_2 + \frac{8}{3\pi}k_2 v_2^2 + \frac{3}{4}k_3 v_2^3 + \cdots \tag{15}$$

$$a^* = \frac{2(v_1 - v_3)}{T} \;=\; \frac{k_1}{m}v_2 + \frac{8}{3\pi}\frac{k_2}{m}v_2^2 + \frac{3}{4}\frac{k_3}{m}v_2^3 + \cdots \tag{16}$$

The equations above are the same as Eqs.(2) and (3) defined in the text. The quantity a^* is not the cylinder's acceleration at the speed v_2. However, by fitting the graph of a^* versus v_2 with a polynomial enables one to determine the force constants k_i. In the next section, it is shown that the force constants determined by this fit are accurate to within one percent for starting angles of 30° or less.

Appendix II: Uncertainties and Accuracy of a^* in determining k_1 and k_2

Some approximations were used in deriving the Eq.(16)

$$a^* = \frac{2(v_1 - v_3)}{T} = \frac{k_1}{m}v_2 + \frac{8}{3\pi}\frac{k_2}{m}v_2^2 + \frac{3}{4}\frac{k_3}{m}v_2^3 + \cdots$$

We examine three of the most relevant ones: a) the small angle approximation, b) the choice of v_2 and $v_2 \approx \frac{v_1+v_3}{2}$, and c) the accuracy for large values of $\frac{k_i}{m}$.

To check these approximations, we choose values for $\frac{k_i}{m}$ and solved the force equation, Eq.(8), numerically to compute the v_i and T_i. We used the Runge-Kutta algorithm with a time step of 10^{-6} seconds and a pendulum length of $l = 2.2m$. From the numerically generated v_i and T_i values, we calculated a^* for 100 periods. The numerically generated a^* versus v data

Table 2: Accuracy of a^*. The values $\frac{k_1}{m} = 0.008\frac{1}{s}$, $\frac{k_2}{m} = 0.05\frac{1}{m}$, and $\frac{k_3}{m} = 0.01\frac{s}{m^2}$ were used for the air friction force. The equation of motion was solved numerically to determine $a^*(v_2)$, where $v_m = v_2$. The function $a^*(v_2)$ was fit with a third order polynomial. The values of $\frac{k_i}{m}$ as determined from fitting $a^*(v_2)$ are listed below as a function of the initial starting angle θ_{max}.

Input Values:	$\frac{k_1}{m} = 0.00800$	$\frac{k_2}{m} = 0.05000$	$\frac{k_3}{m} = 0.01000$
θ_{max}	$\frac{k_1}{m}$ from a^*	$\frac{k_2}{m}$ from a^*	$\frac{k_3}{m}$ from a^*
15°	0.00800	0.05000	0.01000
20°	0.00801	0.04996	0.01003
30°	0.00806	0.04977	0.01016
45°	0.00827	0.04913	0.01050

were fit with a polynomial to determine the $\frac{k_i}{m}$, which are then compared to the "input" values for the force equation of the simulation.

a. Small angle approximation:

The mass of our cylinders ranged from 10 to 100 grams. For this check, we used values of k_i close to a cylinder in our experiment with mass $35.5g$, height $12cm$ and diameter $2.8cm$: $\frac{k_1}{m} = 0.008\frac{1}{s}$ and $\frac{k_2}{m} = 0.05\frac{1}{m}$. We also added a value for $\frac{k_3}{m}$ of $0.01\frac{s}{m^2}$ to check that if there were a cubic term in the frictional force, we would measure it correctly. In Table II we list our results for starting angles of 15°, 20°, 30° and 45°.

As can be seen, the values of $\frac{k_1}{m}$ and $\frac{k_2}{m}$ determined from a^* agree to better than one percent with the input values in the force equation, if the maximum starting angle is 30° or less. The simulation also demonstrates that if there were a k_3 term, it would be measured to within two percent (for starting angles up to 30°).

b. Choice of middle speed:

Another approximation made in the derivation of a^* was to set $v_2 \approx \frac{v_1+v_3}{2}$. To check the accuracy, we compare three different possible choices of the middle speed. For each case, we will use the same values (i.e., the $35.5g$ pendulum) as in the previous table: $\frac{k_1}{m} = 0.008\frac{1}{s}$ and $\frac{k_2}{m} = 0.05\frac{1}{m}$, and a starting angle of 20°. The results are shown in Table 3, where we fitted the curve with only a second order polynomial. Comparison of the simulated results in Table 2 and Table 3 shows that fitting the data with only a second order polynomial instead of a third order yields the same values for $\frac{k_1}{m}$ to within a half percent and $\frac{k_2}{m}$ to one part in 500, supporting our claim that the v^3-term is not needed for the fit.

Table 3 also shows that using v_2 as v_{max} is a better choice than $(v_1+v_3)/2$ for a full cycle or $(v_1+v_2)/2$ for the half cycle. The approximation $v_2 \approx (v_1+v_3)/2$ is a good one, only affecting $\frac{k_1}{m}$ by 3 parts in 800 and $\frac{k_2}{m}$ by one part in 500. If desired, the students could use the formula $a^* = \frac{v_1^2 - v_3^2}{v_2 T}$ instead of $a^* = 2\frac{v_1 - v_3}{T}$. In our lab we use the simpler expression, since the increase in accuracy is negligible.

Table 3: Accuracy of a^*. The values $\frac{k_1}{m} = 0.008\frac{1}{s}$ and $\frac{k_2}{m} = 0.05\frac{1}{m}$ were used for the air friction force. The maximum angle θ_{max} was set at 20°. The equation of motion was solved numerically to determine a^* for different possible values of v_m. The function $a^*(v_m)$ was fit with a second order polynomial. The values of $\frac{k_i}{m}$ as determined from fitting $a^*(v)$ are listed below for three different combinations of v_{max}: $v_{max} = v_2$, $v_{max} = \frac{v_1+v_3}{2}$ and $v_{max} = \frac{v_1+v_2}{2}$.

	Input Values:	$\frac{k_1}{m} = 0.00800\frac{1}{s}$	$\frac{k_2}{m} = 0.0500\frac{1}{m}$
v_{max}	a^*	$\frac{k_1}{m}$ from a^*	$\frac{k_2}{m}$ from a^*
v_2	$2\frac{v_1-v_3}{T}$	0.00804	0.0499
v_2	$\frac{v_1^2-v_3^2}{v_2 T}$	0.00799	0.0500
$\frac{v_1+v_3}{2}$	$2\frac{v_1-v_3}{T}$	0.00814	0.0496
$\frac{v_1+v_2}{2}$	$4\frac{v_1-v_2}{T}$	0.00810	0.0497

c. Strength of Air Friction:

If the strength of the air friction is too large, then the approximations used in deriving a^* might not be valid. To check the range of validity of our approximation, we ran the simulation for different magnitudes of $\frac{k_i}{m}$. The largest values for a^* occur when the mass of the cylinder is small, resulting in a large decrease in v_i. The force constants k_i are proportional to the cylinder's frontal area, and the mass is proportional to the cylinder's volume. This results in the $\frac{k_i}{m}$ values to be proportional to $\frac{1}{d}$. The diameters of our cylinders ranged from 1.4 to 4.0 cm. The values of $\frac{k_i}{m}$ used in the simulation are approximately the experimental values. They vary over a factor of 4 and span the experimental values of our cylinders. In Table 4 we display the results of the simulation. We also list the maximum value of the fractional change in v_i, i.e. $\frac{v_i - v_{i+1}}{v_i}$ after the first half period.

As shown in Table 4, the values of $\frac{k_i}{m}$ determined from a^* are all within one percent of the input values used in the force equation. This is true even when the fractional change in v_i is initially as large as 8.8%.

Table 4: Strength of Air Friction. The accuracy of a^* for different values of $\frac{k_i}{m}$ are shown. The maximum angle θ_{max} was set at 20° and a^* was chosen to be $2\frac{v_3-v_1}{T}$. The values of $\frac{k_i}{m}$ span the range of the cylinders used in this article.

cylinder diameter	Input Values: $\frac{k_1}{m}$ | $\frac{k_2}{m}$	$\frac{k_1}{m}$ from a^*	$\frac{k_2}{m}$ from a^*	Max $\frac{\Delta v_i}{v_i}$
4.0 cm	0.00400 | 0.0250	0.00405	0.0249	0.026
2.8 cm	0.00800 | 0.0500	0.00804	0.0499	0.048
1.4 cm	0.01700 | 0.1000	0.01684	0.1005	0.088

As a final note, we point out that the air friction experiment described here is well suited for an introduction to a reduced chi-square, χ^2, analysis. In particular, if we want to determine if the data is better fit with a v^3 term in the force equation, one can examine the chi-square per degree of freedom, $\frac{\chi^2}{df}$ with and without the cubic term. If the addition of a v^3 term in the force equation does not significantly decrease $\frac{\chi^2}{df}$, it is not needed to describe the data.

To calculate the reduced χ^2, one needs an estimate of the uncertainty in a^*, which is equal to $2(v_1 - v_3)/T$. The period T does not change much during the experiment, however if the cylinder slows down very slowly, i.e. $v_3 \approx v_1$, then the uncertainty in a^* can be large. We demonstrate this in Fig. 6 with a bob made out of steel, which is roughly the same size as as the wooden bob used in Fig. 2, but much more dense.

From the graph in Fig. 6, it can be seen that the uncertainty in a^* increases with a^* and can be treated as roughly proportional to a^*: $\Delta a^* \propto a^* = ea^*$, where e is the fractional error in a^*. Hence, one can express the reduced chi-square as

$$\chi^2 = \sum_{i=1}^{N}[\frac{(a_i^* - (c_1 v_i + c_2 v_i^2 + c_3 v_i^3)}{ea_i^*}]^2$$

$$\frac{\chi^2}{df} = \frac{\chi^2}{N-3}$$

For the linear plus quadratic fit, $c_3 = 0$, and $\frac{\chi^2}{df} = \frac{\chi^2}{N-2}$. For our setup, the

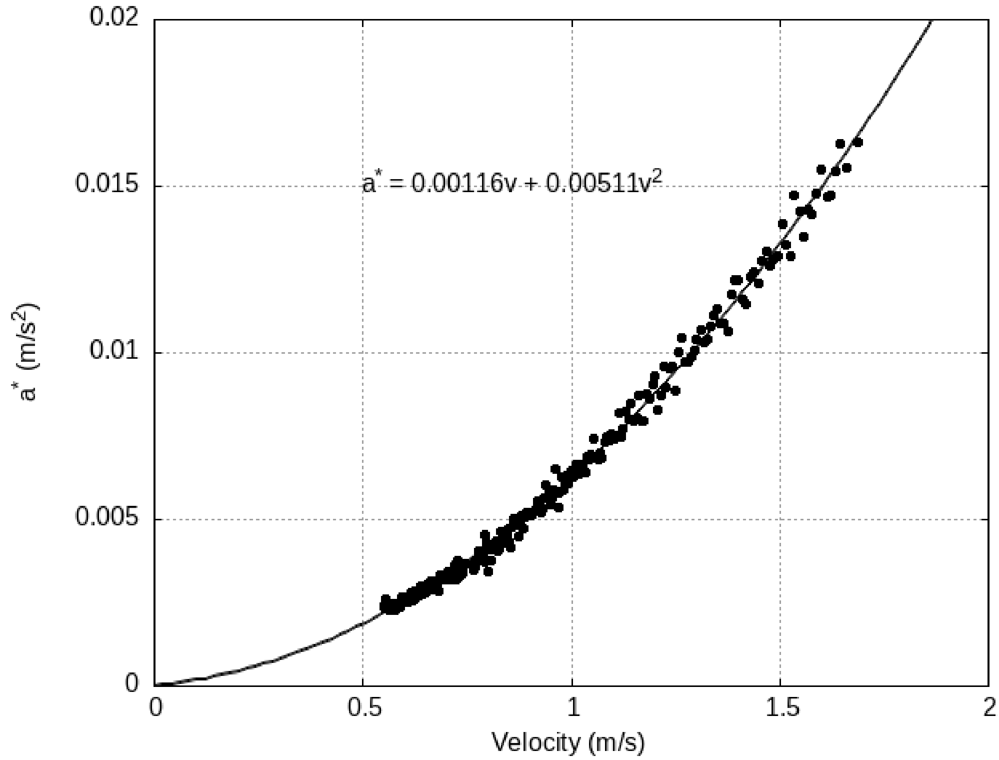

Figure 6: A graph of a^* versus v_2 for a steel bob. The fit plotted uses only linear plus quadratic terms. Adding a cubic term only decreased the $\frac{\chi^2}{df}$ from 1.42 to 1.41, and does not improve the fit.

fractional error e of 0.02 results in a value for $\frac{\chi^2}{df}$ of around 1.4 for either fit. The values of the c_i that minimize the χ^2 can be determined using gnuplot or by writing one's own program, as we did for the steel cylinder. Either way, the students get an introduction to reduced chi-square statistics and to considerations of the quality of a fit.

References

[1] "Oscillations of a quadratically damped pendulum", Carl E Mungan and Trevor C Lipscombe, Eur. J. Phys. **34** (2013) Number 5, 1243-1253

[2] "An analysis of pendulum motion in the presence of quadratic and linear drag", M. Lubarda and V. Lubarda, Eur. J. Phys. **42** (2021) Number 5, 055014

[3] "Vertical fall of a sphere opposed by fluid buoyancy and drag", Aldo Mayme and Carl E Mungan, Eur. J. Phys. **46** (2025) Number 3, 035004

[4] "Introduction to Fluid Mechanics", Robert W. Fox, Alan T. McDonald, John Wiley and Sons, 3rd edition, 1985, p. 455.

[5] "Wind-tunnel blockage effect on drag coefficient of circular cylinders", Anthoine, J.; Olivari, D.; Portugaels, D.; Wind and Structures, Vol.12 (6), 541-551 (2009).

[6] "Design of an automatic pendulum velocity measuring device using light sensors", R. F. Adiati, A. S. Haniyah, A. Kartono, and H. Syafutra, Indonesian Physical Review 8(1), 150-161

[7] "Learning the Resistance Coefficients of Pendulum Motion Using Balls of Various Sizes", Kyung-Ryul Lee and Young-Gu Ju, New Physics Sae Mulli **71** (2021) 12: 1067-1075

[8] "Period-speed analysis of a pendulum", Yavor Kostov, Ragib Morshed, Barbara Hoeling, Ray Chen, and P.B. Siegel, Am. J. Phys. 76, 956-962 (October 2008).

[9] https://en.wikipedia.org/wiki/Drag_coefficient

[10] "Physical Fluid Dynamics", D.J. Tritton, Clarendon Press, 2nd edition, 1988, pp. 21-34, 112.