

Solving the Non-Relativistic Schroedinger Equation for a spherically symmetric potential

If the energy of a particle is non-relativistic, and its interaction is described by a potential energy function, the "physics" is described by solutions to the the time independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V(r)\Psi = E\Psi \quad (1)$$

Whether one is performing a scattering experiment or measuring the bound state energies, will determine the boundary conditions of the solution $\Psi(\vec{r})$.

For bound state solutions, the wavefunction Ψ and the integral $\int \Psi^*\Psi dV$ over all space must be finite. Thus the "boundary conditions" at infinity are: $r \rightarrow \infty$, Ψ must approach zero faster than $1/r$. Since the potential is spherically symmetric, the angular dependence can be separated from the radial. Writing $\Psi = R(r)Y_{lm}(\theta, \phi)$ as a product of a radial part times a *spherical harmonic* ($Y_{lm}(\theta, \Phi)$), the above equation reduces to

$$-\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{l(l+1)}{r^2}R(r)\right) + V(r)R(r) = ER(r) \quad (2)$$

The integer l is related to the particles *orbital angular momentum*. A further simplification is obtained by writing $R(r)$ as $u(r) = R(r)/r$. The radial part of the Schrödinger equation finally becomes the somewhat simple form:

$$-\frac{\hbar^2}{2m}\left(\frac{d^2u(r)}{dr^2} - \frac{l(l+1)}{r^2}u(r)\right) + V(r)u(r) = Eu(r) \quad (3)$$

For Ψ to be finite, $u(0)$ equals 0, and for bound states, $u(r)$ goes to zero as $r \rightarrow \infty$.