Scattering amplitude and partial waves

In assignment 4 you are asked to simulate a scattering experiment. Your simulation will need to have some random error in the data. Before we discuss how to produce a Gaussian probability distribution for the random errors, let's go over the physics of scattering.

The differential cross section, $d\sigma/d\Omega$, can be written in terms of a scattering amplitude, $f(\theta, \phi)$. If the interaction is spherically symmetric, then there is no ϕ dependence for f. So

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \tag{1}$$

where $f(\theta)$ is a complex number with units of length. In our application, $f(\theta)$ will have units of fm^2. For non-relativistic energies, $f(\theta)$ can be determined from the Schroedinger equation with the appropriate scattering boundary conditions at $r = \infty$. If the interaction is spherically symmetric, i.e. $V(\vec{r} = V(r)$, then the Schroedinger equation can be separated into the different orbital angular momentum quantum numbers l. We derived this separation for the first assignment when we treated bound states. We obtained the set of equations:

$$-\frac{\hbar^2}{2m}(\frac{d^2u(r)}{dr^2} - \frac{l(l+1)}{r^2}u(r)) + V(r)u(r) = Eu(r) \tag{2}$$

with an equation for each value of l. The same separation holds for scattering problems. One will obtain a scattering amplitude for each value of orbital angular momentum l, which we label as f_l. The complete scattering solution will have a $Y_{lm}(\theta, \phi)$ added on for each l. For spherical symmetry, where there is no ϕ dependence, so the Y_{lm} reduce to Legendre polynomicals in $cos(\theta)$, $P_l(\theta)$. The scattering amplitude therefore becomes

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)f_l P_l(cos(\theta)) \tag{3}$$

The scattering amplitude $f(\theta)$ and the f_l are complex numbers, with a real and an imaginary part.

The sum over orbital angular momentum l in the expression for the scattering amplitude goes to infinity. The contribution from large l goes to zero, and one only needs to sum over a few values of l. The maximum value needed for l is roughly Rpc, where R is the size of the target and p is the momentum of the projectile. In our application we will only sum over the $l = 0$ and $l = 1$ **"partial waves"**.

One can express the partial wave amplitudes f_l in terms of "phase shifts". The phase shifts are determined from the solution of the Schroedinger equation for each l value. In terms of the **phase shifts**, δ_l, the elastic scattering amplitudes are

$$f_l = \frac{e^{i\delta_l} sin(\delta_l)}{k} \tag{4}$$

where $k = p/\hbar$.

You might wonder where the name phase shift comes from. The δ_l are the shift in phase from the free particle solutions of the Schroedinger equation. In the absence of the potential $V(r)$ ($V(r) = 0$), the solutions to the Schroedinger equation for orbital angular momentum l are the spherical Bessel functions, $j_l(kr)$. In fact, a plane wave traveling in the z-direction expressed in spherical coordinates is given by:

$$e^{ikz} = \sum_{l=0}^{\infty} (2l + 1) i^l j_l(kr) P_l(cos(\theta)) \tag{5}$$

where θ is the angle with respect to the z-axis.

For large r, the spherical Bessel function $j_l(kr) \rightarrow sin(kr - l\pi/2)/(kr)$. The effect of a real potential $V(r)$ is to cause a phase shift in this large r limit: $sin(kr - l\pi/2 + \delta_l)/(kr)$. To solve for the phase shifts δ_l, one just iterates the Schroedinger equation to large r, like we did for the bound state assignment. However for the scattering calculation, the energy is greater than zero, and the solution oscillates. One can obtain the phase shifts by examining the large r solution to the discrete Schroedinger equation. We will not solve the Schroedinger equation for the δ_l in our assignment 4. I'll give you δ_0 and δ_1 for a particular energy, and you will generate simulated data.

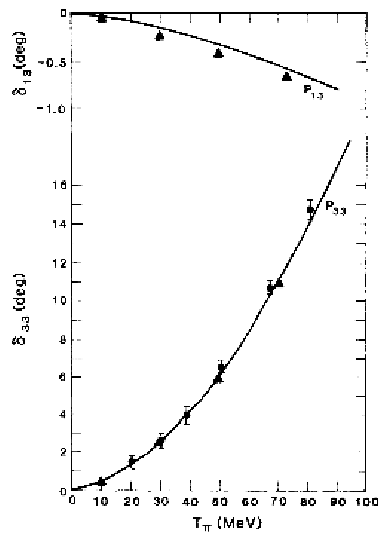

FIG. 3. Spin $\frac{1}{2}$ p-wave pion-nucleon phase shifts. The circles are from Ref. 15, the triangles from Ref. 16.

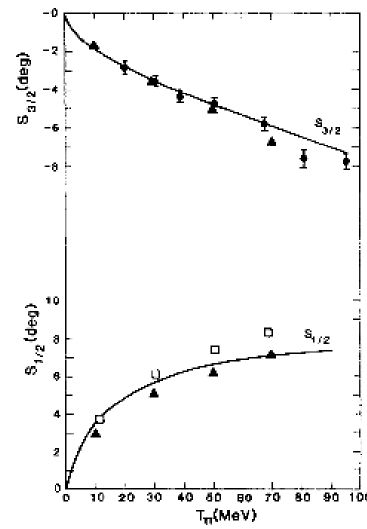

FIG. 5. s-wave pion-nucleon phase shifts. The circles are from Ref. 15, the triangles from Ref. 16, the boxes from Ref. 17.

In the last lecture we came up with the formula for the scattering amplitude in terms of the phase shifts. The result for **elastic scattering** is

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)f_l P_l(cos(\theta)) \tag{6}$$

where

$$f_l = \frac{e^{i\delta_l}sin(\delta_l)}{k} \tag{7}$$

where δ_l are the phase shifts, and $k = p/\hbar$. For the case of elastic scattering the δ_l are real. In our assignment, we only sum up to $l = 1$ since the momentum of the pion is small. In this case, the formula for $f(\theta)$ is

$$f(\theta) = \frac{1}{k}(e^{i\delta_0}sin(\delta_0) + 3e^{i\delta_1}sin(\delta_1)cos(\theta)) \tag{8}$$

Before we go over coding with complex numbers let me discuss where the name phase shift comes from. The δ_l are the shift in phase from the free particle solutions of the Schroedinger equation. In the absence of the potential $V(r)$ ($V(r) = 0$),

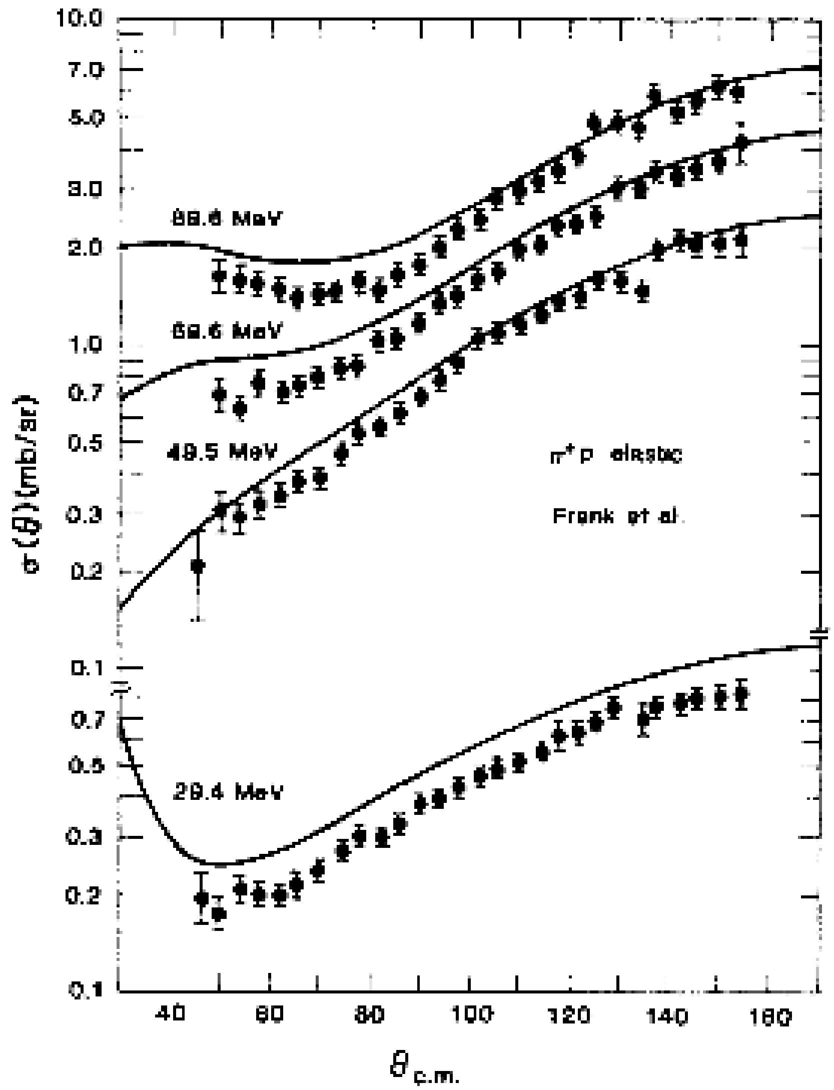

FIG. 7. Comparison with the data of Frank *et al.* (Ref. 19).

the solutions to the Schroedinger equation for orbital angular momentum l are the spherical Bessel functions, $j_l(kr)$. In fact, a plane wave traveling in the z-direction expressed in spherical coordinates is given by:

$$e^{ikz} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(cos(\theta)) \tag{9}$$

where θ is the angle with respect to the z-axis.

For large r, the spherical Bessel function $j_l(kr) \rightarrow sin(kr - l\pi/2)/(kr)$. The effect of a real potential $V(r)$ is to cause a phase shift in this large r limit: $sin(kr - l\pi/2 + \delta_l)/(kr)$. To solve for the phase shifts δ_l, one just iterates the Schroedinger equation to large r, like we did for the bound state assignment. However for the scattering calculation, the energy is greater than zero, and the solution oscillates. One can obtain the phase shifts by examining the large r solution to the discrete Schroedinger equation. We will not solve the Schroedinger equation for the δ_l in our assignment 4. I'll give you δ_0 and δ_1 for a particular energy, and you will generate simulated data.

The sign of the phase shift for elastic scattering at low momentum depends on whether the interaction is attractive or repulsive. For an attractive interaction, the "wave function" curves more than in the absence of an interaction. The result is that the phase of the "wave function" ends up ahead of the "free particle" case. Thus, a **positive phase shift** means that the interaction (for the particular value of l) is **attractive**. For a repulsive interaction, the "wave function" curves less than the free particle case, and the phase shift lags. A **negative phase shift** indicates that the interaction is repulsive. I'll demonstrate this on the board for the $l = 0$ partial wave.

The amplitude is complex, so in your code you will need to use complex numbers where needed. In gcc, one need to include <complex.h>:

```
#include <complex.h>
#include <math.h>
```

You will need to declare any complex variables as complex:

```
complex f;
```

Some commands that you might need are cexp(), which is the complex exponential function, and cabs(), which is the complex absolute value squared. For example:

$$
\begin{aligned}
cexp(x) &\rightarrow e^x \\
cabs(f) &\rightarrow \sqrt{f^* f}
\end{aligned}
$$

Also, you will need to find the center of mass momentum, since $k = p_{cm}/\hbar$. Now you are ready to write your code for assignment 4. I'd recommend to first produce the data without any "Gaussian" scatter. Then throw the Gaussian dice to simulate the statistical error in a real experiment for each data point.