Born Approximation and Numerical Integration

Often in physics problems one needs to evaluate a definate integral. If it is not possible to find the "anti-derivative" of the integrand then numerical methods may be the only way to solve the problem. For our next assignment we will calculate an integral that will give us an approximation to elastic scattering. The particular problem we will do is to solve for K^+ particles scattering off of nuclei. First I will go over a simple way to do numerical integration for one variable. Then, we will talk about the physics involved in scattering experiments, and an approximate method for calculating cross sections: the Born Approximation.

Numerical Integration

Rectangle and Trapazoid Rules

Consider the definate integral, $\int_a^b f(x)\,dx$. The integral is equal to the area under the curve $f(x)$ from a to b. Our task is then to estimate the area under this curve. The simplest way is to divide the total area under the curve into small rectangles and add up the areas of each small rectangle. Let the base of each rectangle be on the x-axis and the height be the value $f(x)$. We divide the x-axis into small segments of equal length h. Then the sum of the areas of the rectangles is given by

$$\begin{aligned}\int_a^b f(x)\,dx &\approx hf(a) + hf(a+h) + hf(a+2h) + \cdots + hf(b-h) \\ &\approx h(f(a) + f(a+h) + f(a+2h) + \cdots + f(b-h))\end{aligned}$$

The smaller that h is, the better the approximation. We can write this expression in summation notation as follows:

$$\int_a^b \approx \sum_{n=0}^{N-1} h\ f(a+nh) \tag{1}$$

where n is an integer and $h = (b-a)/N$. The larger that N is, the more accurate is the sum.

One can do slightly better by using trapazoids instead of rectangles. The area of a trapazoid with a base h, a left height of $f(a)$ and a right height of $f(a+h)$ is $h(f(a)+f(a+h))/2$. Using trapazoids instead of rectangles yields

$$\begin{aligned}
\int_a^b f(x)\,dx &\approx h\frac{f(a)+f(a+h)}{2} + h\frac{f(a+h)+f(a+2h)}{2} + \cdots \\
&\approx h(\frac{f(a)}{2} + f(a+h) + f(a+2h) + \cdots + f(a+(N-1)h) + \frac{f(b)}{2}) \\
\int_a^b f(x)\,dx &\approx h(\frac{f(a)}{2} + \sum_{n=1}^{N-1} h\ f(a+nh) + \frac{f(b)}{2})
\end{aligned}$$

where n is an integer and $h = (b-a)/h$. As before, the approximation gets better as $N \to \infty$. Also note that this expression is exact if $f(x)$ is a line.

There are other algorithms that are used to carry out numerical integation. At the end of these notes I have summarize another simple one, Simpson's rule. If time allows, I'll summarize the method of Gaussian Quadrature. Now, lets go over the integral in your assignment and discuss the physics behind the integral.

```c
//Demonstration program for rectangle integration
//

#include <stdio.h>
#include <math.h>
#include <stdlib.h>

int main ()
{
   double x,sum, dx, pi;
   int i,n;

        pi=3.1416;
        printf("Input the number of points: \n");
        scanf("%d", &n);
        sum=0.0;
        dx=pi/n;
        for (i=1; i<n; i++)
         {
          x=i*dx;
          sum=sum+sin(x);
         }
//We don't need to add the endpoint values, since sin(0)=sin(pi)=0.
        sum=sum*dx;
        printf("The integral is %f \n",sum);
        return(0);
}
```

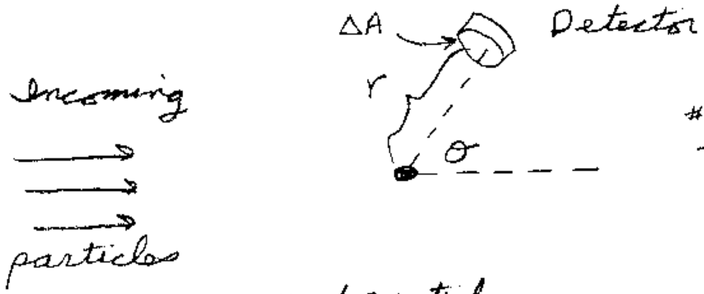
Scattering Experiments

Scattering experiments are often carried out to understand the interaction between elementary particles as well as information on the excited states of nuclei and particles. Basically, one uses a beam of particles, all with the same momentum, that "hits" a target. One can also have colliding beams. One measures the probability that a particle in the beam will be scattered off at a particular angle. Let's define some terms:

Incident Flux: The number of particles per area per time.

Solid Angle: If the detector has an area A and is a distance r from the target, then the solid angle is $\Delta\Omega \equiv A/r^2$. The full solid angle, a spherical shell, is $4\pi r^2/r^2 = 4\pi$.

Scattering Cross Section



$$\frac{\text{\# of particles detected}}{\text{sec}}$$

Flux $J \equiv \dfrac{\text{\# of particles}}{\text{area sec}}$

$$\frac{\text{\# of particles detected}}{\text{sec}} \sim J\left(\frac{\Delta A}{r^2}\right)$$

$$= \sigma(\theta,\varphi)\ J\ \Delta\Omega$$

$$\sigma(\theta,\varphi) = \frac{\text{\# of particles detected / sec}}{J \quad \Delta A}$$

differential cross section ↑

$$\sigma_{TOTAL} = \iint \sigma(\theta,\varphi)\, d\Omega$$

Differential Cross section: The number of particles detected per sec per incident flux per solid angle. It is given the symbol $\sigma(\theta, \phi)$, or $(d\sigma)/(d\Omega)$. For unpolarized targets and beams, the differential cross section will only depend on θ, the scattering angle, and is labeled as $\sigma(\theta)$

Total Cross section: The total number of particles scattered. The total cross section σ equals the differential cross section integrated over all solid angle: $\sigma_{total} = \int \sigma(\theta, \phi) d\Omega$.

A graph of the differential cross section for K^+ particles scattering off ^{12}C is given on the next page. The units for the differential cross section are Area/(solid angle), and the units for the total cross section is Area. The area represented by the total cross section can be thought of as follows: It is the effective area per target such that if the particle "hits" this area it is scattered.

The cross sectional area of a nucleon is around $\pi r^2 \approx \pi\ fm^2$, since the nucleon radius is around one Fermi. The nucleon's cross sectional area is thus around $10^{-30}\ fm^2$ or $10^{-26}\ cm^2$. The unit for area in nuclear and particle physics is the barn, which is defined as $1\ barn \equiv 10^{-24}\ cm^2 = 100\ fm^2$. A barn is around 100 times the area of a nucleon, and is quite large on the nuclear scale. Note that the units on the graph for the differential cross section, $\sigma(\theta)$ are milli-barns per ster-radian (mb/sr).

Calculating the Cross Section

If the interaction between the incident particle and the target can be treated non-relativistically and be represented by a potential energy function $V(r)$, then the scattering cross section can be calculated using the Schroedinger equation. For our assignment we will use an approximation to the differential cross section that only requires the computation of an integral. The approximation is called "the Born approximation". If you are interested in how it is derived, you can look at my previous notes or an undergraduate quantum text. At the heart of the approximation is the evaluation of the following integral:

$$\int_0^\infty V(r) r sin(qr) dr \tag{2}$$

where we will take $V(r) = V_0/(1 + e^{(r-R)/a})$. The quantity $q = p/\hbar$ and represents the momentum/$\hbar$ that is transfered to the target. In terms of the kaon's momentum p and the scattering angle, θ, $q = 2pc/(\hbar c) sin(\theta/2)$. As before, we will take the size of the potential to be the size of the nucleus, so $R = 1.28 A^{1/3}\ fm$. We will take $a = 0.6\ fm$. The cross section is most easily expressed in terms of the scattring amplitude, $f(\theta)$. The Born approximation gives:

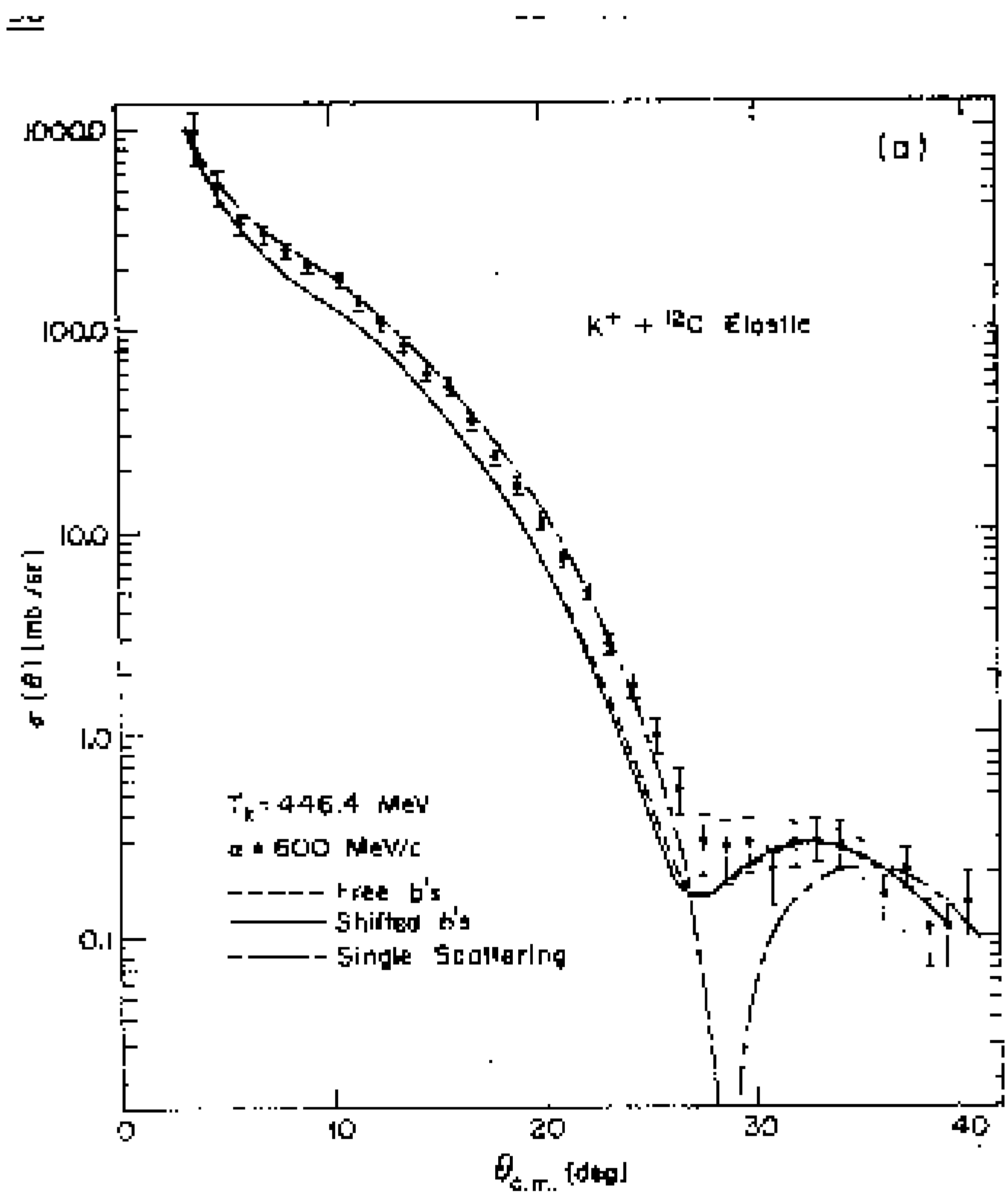
$$f(\theta) = -\frac{1}{\hbar c}\frac{mc^2}{pc}\frac{1}{sin(\theta/2)}\int_0^\infty V(r) r sin(qr) dr \tag{3}$$

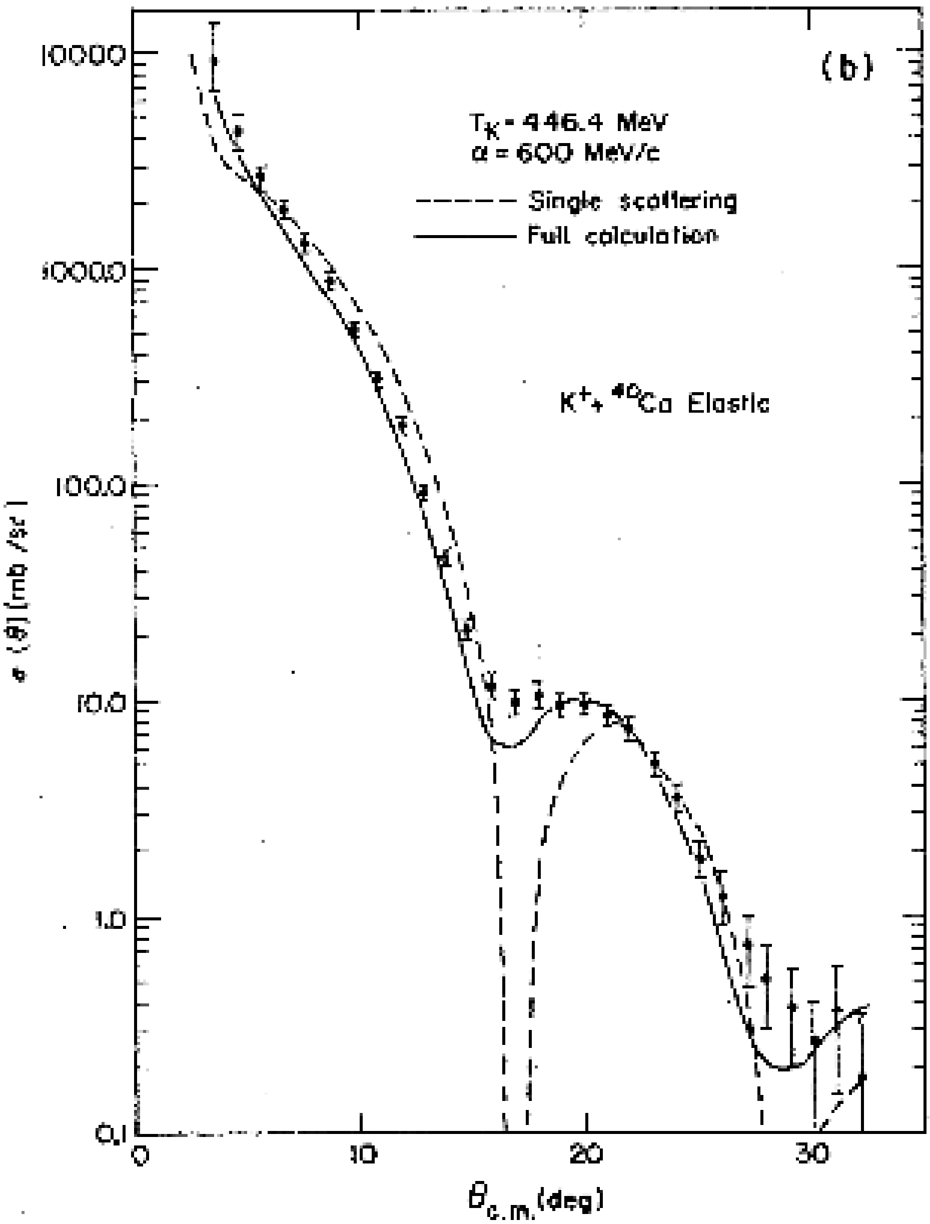
for the scattering amplitude. In terms of the scattering amplitude, the differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \tag{4}$$

Putting everything together, the differential cross section is

$$\sigma(\theta) = |-\frac{1}{\hbar c}\frac{mc^2}{pc}\frac{1}{sin(\theta/2)}\int_0^\infty V(r) r sin(qr) dr|^2 \tag{5}$$


(a)
K$^+$ + ^{12}C Elastic
T_K = 446.4 MeV
α = 600 MeV/c
Free b's
Shifted b's
Single Scattering
σ (θ) [mb/sr]
θ$_{c.m.}$ (deg)
1000.0
100.0
10.0
1.0
0.1
0
10
20
30
40


(b)
T_K = 446.4 MeV
α = 600 MeV/c
Single scattering
Full calculation
K⁺+ ⁴⁰Ca Elastic
σ (θ) [mb /sr]
θ_c.m.(deg)
10000
1000.0
100.0
10.0
1.0
0.1
0
10
20
30

The Born Approximation

The Born approximation can be derived from time dependent perturbation theory. The probability for a transition to occur from an initial state $\psi_i \propto e^{ik_i \cdot r}$ to a final state $\psi_f \propto e^{ik_f \cdot r}$ (where $k = p/\hbar$) is proportional to the matrix element

$$< \psi_f|V|\psi_i > \propto \int_0^\infty V(r) e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}} d^3\vec{r} \tag{6}$$

This integral, which is the Fourier transform of V(r) with respect to the momentum transfer, produces a scattering diffraction pattern. The kinematics enters the problem only with the momentum transfer, which is most conveniently defined as $\vec{q} = \vec{k}_f - \vec{k}_i = (\vec{p}_f - \vec{p}_i)/\hbar$.

$$< \psi_f|V|\psi_i > \propto \int_0^\infty V(r) e^{i\vec{q}\cdot\vec{r}} d^3\vec{r} \tag{7}$$

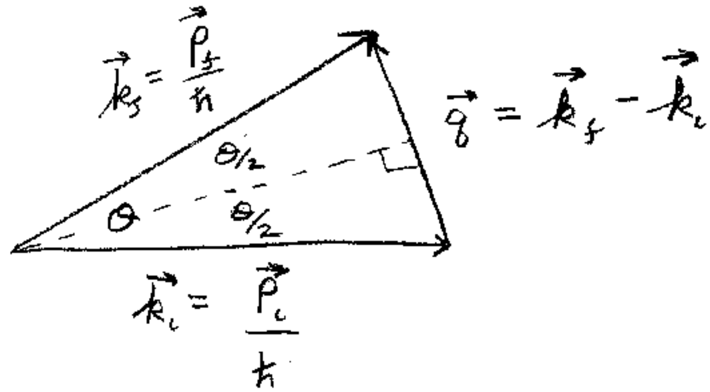
At first glance, the integral looks difficult. However, if we choose the coordinate system for the integral such that the z-axis is directed along $\vec{q}$, then $\vec{q}\cdot\vec{r} = |q|\ |r|\ cos(\alpha) = qr\ cos(\alpha)$, where α is the angle between $\vec{q}$ and $\vec{r}$. With this choice of coordinate system, the integral becomes:

$$\int_0^\infty V(r) e^{i\vec{q}\cdot\vec{r}} d^3\vec{r} = \int_0^\infty V(r) \int_0^\pi e^{iqrcos(\alpha)} 2\pi sin(\alpha)\ d\alpha r^2 dr \tag{8}$$

The angular part of the integral can be evaluated since $sin(\alpha)$ is the derivative of $cos(\alpha)$:

$$\begin{aligned}
\int_0^\infty V(r) e^{iqrcos(\alpha)} 2\pi sin(\alpha) r^2\ d\alpha dr &= \int_0^\infty V(r) 2\pi \left(\frac{-e^{iqrcos(\alpha)}}{iqr}\right)\Big|_0^\pi r^2 dr \\
&= \int V(r) 2\pi\ \frac{e^{iqr} - e^{-iqr}}{iqr}\ r^2 dr \\
&= \frac{4\pi}{q} \int_0^\infty V(r) r\ sin(qr) dr
\end{aligned}$$

which is the integral you need to solve numerically.



$$\frac{|\vec{q}|}{2} = |\vec{k}_i| \sin \theta/2$$

$$q = 2\,|\vec{k}_i| \sin \theta/2$$

$$q = \frac{2p}{\hbar} \sin \theta/2$$

$$\sigma(\theta) = |f(\theta)|^2$$

where $f(\theta) = -\left(\frac{1}{\hbar c}\right)\left(\frac{mc^2}{pc}\right)\frac{1}{\sin\theta/2}\int_0^\infty V(r)\, r \sin qr \, dr$

Note that the angle enters through q, since $q = (2p/\hbar) sin(\theta/2)$. When integrating over r, sin(qr) can change sign and the integral can be zero. That is, there can be angles where this integral will be zero. The cancelation in the integral is in essence destructive interference from the particle scattering off different parts of the target. One sees the same effect in double slit interference. As with the double slit interference pattern, the interference occurs at smaller angles for larger targets. This effect is seen in the comparison of the K^+ scattering data off ^{12}C versus ^{40}Ca. The diffraction interference occurs at around 28° for ^{12}C and around 17° for ^{40}Ca for kaons with the same momentum.

If $V(r)$ is proportional to the density of the target, this integral is often refered to as the **form factor** of the interaction.

Simpson's Rule

The trapaziod rule used two values of $f(x)$ for each interval, and is exact if $f(x)$ is a line. One can do a little bit better using equal spacing and three values of $f(x)$. We will derive in class the following formula for the area under a parabola using values of the function evaluated at $x - h$, x, and $x + h$:

$$AREA = h(\frac{f(x-h)}{3} + \frac{4f(x)}{3} + \frac{f(x+h)}{3}) \tag{9}$$

We can do the same treatment with parabola fits that we did with trapazoids. We divide the interval from a to b into N equal segments. For each consequitive triplet of segments we can use the parabola formula above. The sum over all the segments is:

$$\begin{aligned}\int_a^b f(x)\,dx \quad \approx \quad & h((\frac{f(a)}{3} + \frac{4f(a+h)}{3} + \frac{f(a+2h)}{3}) + \\ & (\frac{f(a+2h)}{3} + \frac{4f(a+3h)}{3} + \frac{f(a+4h)}{3}) + \\ & (\frac{f(a+4h)}{3} + \frac{4f(a+5h)}{3} + \frac{f(a+6h)}{3}) + \cdots)\end{aligned}$$

Notice that the even increments of h are counted twice resulting in a 2/3 factor on even multiples of h:

$$\begin{aligned}\int_a^b f(x)\,dx \quad \approx \quad & h(\frac{f(a)}{3} + \frac{4f(a+h)}{3} + \frac{2f(a+2h)}{3} + \\ & \frac{4f(a+3h)}{3} + \frac{2f(a+4h)}{3} + \frac{4f(a+5h)}{3} + \\ & \cdots + \frac{4f(a+(N-1)h)}{3} + \frac{2f(b)}{3})\end{aligned}$$

where $h = (b - a)/N$ and N must be an even number. N must be even so that the function is evaluated at an odd number of points. This is to insure that the triplet of evenly spaced segments fits properly into the interval between a and b.

The parabolic formula for numerical integration is called Simpson's Rule. It is exact if $f(x)$ is a parabola.