

Statistics of Nuclear Decay

The "radioactivity" of a radioactive sample decreases in time. By radioactivity we mean the number of times a radioactive decay occurs in the sample per unit time. A standard unit of activity is decays per second. In our experiments in the laboratory we will observe that the radioactivity decreases in a very simple manner, by a "half-life" formula. In this chapter, we will present the half-life formula, discuss the physics behind the formula, and derive an expression for the statistical uncertainty for our measurements of activity. These three ideas derive from one principle.

"Half-life decay formula"

The number of radioactive nuclei and the activity of a sample decrease in time in a very simple way, via a "half-life formula". Let $N(t)$ be the number of radioactive nuclei in a sample. There exists a particular time τ , called the half-life, such that $N(t)$ decreases by a factor of $1/2$ after a time τ , for any time t . In other words: $N(t + \tau) = N(t)/2$ for any time t . For example, suppose initially we start with 64 radioactive nuclei. After a time τ we have 32 radioactive nuclei. After a time 2τ we have 16 radioactive nuclei, etc. The table below summarizes the results:

Time	$N(t)$
0	64
τ	$32 = 64(1/2)^1$
2τ	$16 = 64(1/2)^2$
3τ	$8 = 64(1/2)^3$
4τ	$4 = 64(1/2)^4$

From the right column, one can deduce a simple formula for $N(t)$:

$$N(t) = N_0 \left(\frac{1}{2}\right)^{t/\tau} \quad (1)$$

where N_0 is the number of radioactive nuclei at time $t = 0$. The formula holds for our simple example, and also satisfies $N(t + \tau) = N(t)/2$ for any time t . In fact the formula is valid for any time t , even for times between multiples of τ .

As previously mentioned, the activity of a sample is the number of decays/time. Units for activity that we will use in this class are:

$$\begin{aligned} 1 \text{ Curie} &= 3.7 \times 10^{10} \text{ decays/sec} \\ 1 \text{ milli-Curie} &= 1 \text{ mCi} = 3.7 \times 10^7 \text{ decays/sec} \\ 1 \text{ micro-Curie} &= 1 \mu\text{Ci} = 37,000 \text{ decays/sec} \\ 1 \text{ Bequerel} &= 1 \text{ Bq} = 1 \text{ decay/sec} \end{aligned}$$

Samples with an activity of a Curie and a milli-Curie is quite hazardous. In our laboratory, our samples will be a few micro-Curies. Since the activity $A(t)$ equals $-\frac{dN}{dt}$, the activity of a sample also follows the simple half-life formula:

$$A(t) = A_0 \left(\frac{1}{2}\right)^{t/\tau} \quad (2)$$

where A_0 is the initial activity, $A(t = 0)$, of the sample.

The half-life τ depends on the particular radioactive nucleus. For example, the radioisotope ^{137}Cs has a half-life of 30 years. The radioisotope ^{127}Ag has a half-life of 200 seconds, and ^{238}U a half-life of around 4.47×10^9 years. Half-lives can be as long as a billion years, or as short as a fraction of a second. To find the half-life of a particular isotope, you need to look up the value in an appropriate table. I have links on my home page to sites that have extensive data on nuclei.

The "Physics" behind the half-life formula

Why does the activity and the number of radioactive nuclei in a sample decrease as the "half-life" formula? The answer can be understood from the physics of quantum mechanics:

A radioactive nucleus has a certain probability per unit time to decay.

The probability to decay/time is termed the "decay constant", and is given the symbol λ . The value of the decay constant depends on the nature of the particular decay process.

$\lambda \equiv$ the probability to decay per unit time (units of 1/time)

From this assumption, one can "derive" the half-life decay rule as follows. Letting $N(t)$ stand for the number of radioactive nuclei in the sample at time t , the number of nuclei that decay in a time Δt equals $N(t)$ times the probability that one will decay in the time Δt . Stated in an equation, we have:

$$N(t) - N(t + \Delta t) = N(t)\lambda(\Delta t) \quad (3)$$

Dividing by Δt , and multiplying by -1 gives:

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} = -\lambda\Delta t \quad (4)$$

If we take the limit as Δt goes to zero, we obtain a simple differential equation for $N(t)$:

$$\frac{dN}{dt} = -\lambda N \quad (5)$$

This is a fairly simple differential equation, whose solution is a decaying exponential function:

$$N(t) = N_0 e^{-\lambda t} \quad (6)$$

where N_0 is the number of radioactive nuclei at time $t = 0$. This exponential decay is the same as the half-life decay formula given above. If we change from base e to base 2, we can see this. Since $2 = e^{\ln 2}$, we have $e = 2^{1/(\ln 2)}$. If we substitute this expression for e into the above equation, we obtain:

$$N(t) = N_0 2^{-(\lambda t)/(\ln 2)} = N_0 \left(\frac{1}{2}\right)^{(\lambda t)/(\ln 2)} \quad (7)$$

Comparing this expression with the half-life formula, we see that they are identical and that the half-life is related to the decay constant by:

$$\tau = \frac{\ln 2}{\lambda} \quad (8)$$

This is an important relationship. It allows us to determine the decay constant by measuring the half-life.

Lets get a feeling for the numbers involved for one of our classroom isotopes. Consider ^{137}Cs , which has a half-life of 30 years. To calculate the decay constant, λ , in units of 1/min we just substitute into the above equation:

$$\lambda = \frac{\ln 2}{\tau} = \frac{\ln 2}{30 \text{ years} (365 \text{ days/year}) (24 \text{ hrs/day}) (60 \text{ min/hr})} = 4.4 \times 10^{-8} \frac{1}{\text{min}} \quad (9)$$

This means, that if you wait one minute, the probability that a particular radioactive Cs^{137} nucleus will decay is only 4.4×10^{-8} , or 1 out of 23 million. That is about the same chance as winning the state lottery! If the chance is so small that a nucleus will decay, then how come we have so many decays in one minute? Our Geiger counter reads a few thousand counts per minute. The reason is that there are many many radioactive nuclei in the sample. Although each one has a tiny chance to decay, there are so many nuclei in the sample that a relative large number actually decay each minute. To determine how many radioactive nuclei are in the sample, we need the relationship between activity and the decay constant. This relationship can be determined as follows:

The activity, $A(t)$, is the number of decays per unit time, or decays/sec. Consider measuring for a time Δt . In terms of the number of radioactive nuclei at time t , $N(t)$, we have:

$$A(t) = \frac{N(t) - N(t + \Delta t)}{\Delta t} \quad (10)$$

Taking the limit as Δt goes to zero, gives:

$$A(t) = -\frac{dN}{dt} = -\frac{d(N_0 e^{-\lambda t})}{dt} = \lambda N(t) \quad (11)$$

This relationship is often written as:

$$A = \lambda N = \frac{N \ln 2}{\tau} \quad (12)$$

This is a very important relationship, and we will use it in our laboratory experiments. With our knowledge of the half-life of an isotope, we can measure the activity of a sample and determine the number of radioactive nuclei. For example, consider ^{137}Cs . For a one microcurie ^{137}Cs sample, the number of radioactive nuclei can be calculated. One microcurie is an activity of 37,000 decays per second, which means that $A = 37,000$ (1/sec). We need to express the half-life, which is 30 years, in seconds. $\tau = (30\text{years})(365\text{days/year})(24\text{hrs/day})(3600\text{sec/hr}) = 9.46 \times 10^8$ seconds. To solve for N , we just plug into the above equation:

$$N = \frac{A\tau}{\ln 2} = \frac{37000(9.46 \times 10^8)}{\ln 2} \approx 5 \times 10^{13} \quad (13)$$

This might seem like a large number of nuclei, I wouldn't mind having this many pennies. However, compared to the total number of nuclei in the sample, this is a very small number. The sample contains around 10^{22} nuclei, close to Avagadro's number. We can also determine the number of grams of ^{137}Cs that are in the sample using Avagadro's number: $m = 137(5 \times 10^{13}) / (6.02 \times 10^{23}) = 1.15 \times 10^{-8}$ grams.

The discussion in this section gives us a good physical picture of what is taking place inside a sample which contains a radioactive isotope. Usually there are a large number of radioactive nuclei, each with a small chance of decaying per second. Every second, a certain number "win the lottery" and decay to a lower energy state. Although there are a large number of radioactive nuclei, this number is very small compared to the total number of nuclei(atoms) in the sample. In biological experiments with radioisotopes, one wants to use the smallest amount of isotope as possible.

Since it is fairly easy to detect most decays, we can limit the amount we use to a small value.

On philosophical grounds, it is quite disturbing that there exists a randomness to nature. The picture we have from quantum mechanics is that we can never know exactly when a particular nuclei will decay. The best we can do is to calculate the probability to decay per sec, the decay constant λ . As humans we are not completely helpless, but being limited to only calculate a "probability to decay" is disturbing. There is a famous quote of Einstein which goes "God does not play dice with the universe". However, all experiments of modern physics make us believe the contrary. When we do the half-life experiment in this class and verify the half-life formula, we are indirectly verifying the probabilistic nature of radioactive decay. We are watching nature toss the dice. Perhaps in the future we will have a more satisfying theory of how nature works.

The relationship between half-life and the decay constant does make sense: $\tau = (\ln 2)/\lambda$. If λ is large, τ is small, and visa-versa. That means that if a decay process has a high probability to occur, then the half-life will be small. Processes that have a small probability to occur will have a long half-life. Having an understanding of the probabilistic nature of the decay helps us determine the statistical uncertainties of our isotope measurements. In the next section, we examine the statistical uncertainties in nuclear isotope measurements.

Statistical Uncertainties

When you measure radioactive decay, you usually measure the number of recorded counts per unit time. If you were to repeat the measurement, you would obtain (most likely) a different number. This is because the decay process is probabilistic. Since we understand the statistical nature of the decay, we can estimate the variation in the number of recorded counts. This variation is called the statistical uncertainty of a measurement. It is an uncertainty that is well understood.

Consider a measurement that is done for a time period Δt . Suppose that there are N radioactive nuclei, and that each one has a probability p to decay in the time Δt . Since λ is the probability to decay per time, $\lambda\Delta t$ is the probability to decay in the time interval Δt :

$$p = \lambda\Delta t \tag{14}$$

The probability q that one of the radioactive nuclei does not decay is then just,

$$q = 1 - p \tag{15}$$

If there are N radioactive nuclei, and each one has a probability p to decay, then on the average there will be pN nuclei that decay in the time Δt . So the average number that decay is given by:

$$\bar{n} = pN \tag{16}$$

Note that if p is small, N does not change significantly from one measurement interval to the next. When we make a measurement of the number of decays, we don't always measure the average pN . Since the decay process is probabilistic, we usually get more or less than the average. To calculate the variability of the number of particles detected, we consider the function $F(n)$, which represents the probability that n nuclei decay out of the N radioactive nuclei present in the sample.

$F(n)$ = the probability that n nuclei decay (out of a total of N radioactive nuclei) in the time interval Δt .

We would like to know this function $F_N(n)$, because then we can calculate how much it can vary about its average value. $F_N(n)$ can be determined from the Binomial theorem. If p is the probability to decay, and q the probability not to decay, then the chance that n decays will occur in a particular ordering is $p^n q^{(N-n)}$. The number of orderings that can occur are n choose N . So $F_N(n)$ is given by:

$$F_N(n) = \frac{N!}{(N-n)!n!} p^n q^{N-n} \tag{17}$$

The sum on the right is just the binomial expansion of $(p+q)^N$, so the $\sum_{n=0}^{n=N} F_N(n) = (p+q)^N$. Thus, this sum equals one if $p+q = 1$ as it should. To find the average value for n , we need to compute the following sum:

$$\bar{n} = \sum_{n=0}^{n=N} \frac{nN!}{(N-n)!n!} p^n q^{N-n} \tag{18}$$

which should be equal to pN . See Appendix I for a tricky derivation of this relationship: $\bar{n} = pN$.

Next, we want to quantify how much our measurements of n will vary. We can define the standard deviation, σ , or root-mean-square as the average of the squares of the differences from the average. In terms of our expression for $F_N(n)$, the standard deviation is given by:

$$\sigma^2 = \sum_{n=0}^{n=N} (n - \bar{n})^2 F_N(n) \quad (19)$$

Substituting our expression for $F(n)$ gives:

$$\sigma^2 = \sum_{n=0}^{n=N} (n - \bar{n})^2 \frac{N!}{(N-n)!n!} p^n q^{N-n} \quad (20)$$

After some algebra using the same "p and q" trick, one obtains that

$$\sigma^2 = Npq \quad (21)$$

the standard deviation is the square-root of (Npq) . This expression for the standard deviation, \sqrt{Npq} , is an estimate of how much we expect our measurements to vary about the average value of Np . For radioactive decay, we have a very nice situation, because p is usually very small. For ^{137}Cs , p was around 4.4×10^{-8} . Thus, $q = 1 - p$ is very very close to one! So our formula for the standard deviation is $\sigma \approx \sqrt{pN}$. This is actually a very good approximation. However, Np is just the average number of counts, \bar{n} , so we are left with the simple formula for the standard deviation of our measurements:

$$\sigma \approx \sqrt{\bar{n}} \quad (22)$$

WoW!! This is a very important result, and we will use it in our laboratory experiments. It states that the standard deviation of our measurements is just the square-root of the average number of counts. We don't need to know N or p , so long as $p \ll 1$. We will use this result two different ways in our laboratory class: 1) to check if our detector is operating properly, and 2) to estimate the statistical uncertainty in our measurements.

1. Checking if a detector is operating properly

If one takes successive measurements with a properly operating detector, the standard deviation of the values recorded should be the square-root of the average of the values. For example, if one takes M one minute readings of a radioactive sample. Let the number of particles detected be n_i for the i 'th measurement. For these M values, one can calculate the average value of the n_i :

$$\bar{n}_{exp} = \frac{\sum_{i=1}^M n_i}{M} \quad (23)$$

One can also calculate the experimental value of the standard deviation:

$$\sigma_{exp} = \sqrt{\frac{\sum_{i=1}^M (n - n_i)^2}{M - 1}} \quad (24)$$

If the detector is operating properly, then σ_{exp} should be approximately the square-root of \bar{n}_{exp} . How close is close enough? One can define a "reliability factor" R:

$$R = \frac{\sigma_{exp}}{\sqrt{\bar{n}_{exp}}} \quad (25)$$

If the detector is operating properly, then R should be close to one. From a statistical analysis, it can be shown that

Value of R	Probability
$0.82 < R < 1.18$	80 % of the time
$0.78 < R < 1.21$	90 % of the time
$0.70 < R < 1.30$	98 % of the time

If we obtain a value for R that is less than 0.7 or greater than 1.3, then most likely our detector is not working properly. It is interesting to note that R cannot be too small. You might think that your detector is really good if the readings are all close to each other every time you take a measurement. This line of reasoning is false. The measurements must have a standard deviation equal to the square-root of the average value.

2. Estimating statistical uncertainties of the counts recorded

In a radiation "counting" measurement, one does not repeat the measurement many times to get an average and an uncertainty. Instead, one makes one measurement and assumes that the amount of radiation detected is approximately the average value that one would get by averaging over many measurements. This being the case, then the standard deviation is given by the square-root of the number of counts recorded. That is, if one measurement is taken and n counts are recorded by the detector, then the uncertainty in this measurement is the square-root of n:

$$Counts\ Recorded = n \pm \sqrt{n} \quad (26)$$

The above result is extremely important, and we will use it in our laboratory experiments. If we record n total counts in an experiment, then we know that the statistical uncertainty is the square-root of n. The square-root of n is referred to as the statistical uncertainty. It is not an uncertainty, or error, due to the poor quality of our

detector. It is an uncertainty due to the statistical nature of the decay process. There is nothing we can do to reduce this uncertainty. It is very important to realize that n stands for the total number of counts recorded, and is not the number recorded per unit time. In the above equation, **n is unitless!** Only take the square-root of a unitless number. Some examples may clarify the significance of statistical uncertainty.

Suppose we collect data for 10 minutes and record 102 counts. Then the statistical uncertainty of our measurement is $\sqrt{102} = 10.1$. We would express our measurement for the 10 minutes as 102 ± 10 counts, since we want to keep the correct number of significant figures. If we want to express our measurement in terms of counts/min, we have $(102 \pm 10)/10$ min, which is equal to 10.2 ± 1.0 counts/min.

Suppose now we do the same measurement, but collect data for 20 minutes. Suppose we record a total of 195 counts. Now, our uncertainty is $\sqrt{195} = 14$ to two significant figures. For this 20 minute measurement, the number of counts we recorded is 195 ± 14 counts. In terms of counts per minute, we have $(195 \pm 14)/20$ min, which is equal to 9.75 ± 0.70 counts/min. Now, if we compare with our measurement for 10 minutes, we see that our total uncertainty is greater, $14 > 10$, since the total number of counts recorded is greater.. However, the uncertainty in terms of counts/min is less with the 20 minute count, $0.7 < 1.0$. By recording counts for a longer period of time we can reduce the *fractional statistical uncertainty*. The fractional statistical uncertainty is easy to calculate, since we know the uncertainty is the square root of the total number of counts n :

$$\text{fractional statistical error} = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \quad (27)$$

The above formula is very useful in determining how long we need to run our counting equipment. For example, suppose we want to know the counting rate to an accuracy of 5%. For the statistical error to be this small, we must have $n = 400$ counts. If our counting rate is approximately 40 counts/min, we must record data for at least 10 minutes to reduce the statistical uncertainty to under 5%.

Many of the basic ideas for our experimental work were discussed in this chapter: the half-life formula, the relationship between half-life and the decay constant and their relationship to the number of radioactive nuclei. The formula for the statistical error is very important and will be used extensively in the laboratory. The formulas all derive from the probabilistic nature of radioactive decay.

Appendix I: A Derivation of $\bar{n} = pN$

In this appendix we derive the simple formula for the average number of counts, \bar{n} . The average number of counts is given by:

$$\bar{n} = \sum_{n=0}^{n=N} \frac{nN!}{(N-n)!n!} p^n q^{N-n} \quad (28)$$

Consider the right side of the equation. The sum depends on N , p , and q . N is always independent of p and q . Let's suppose that **the variables p and q are independent of each other**. Then the right side of the equation can be written as

$$\sum_{n=0}^{n=N} \frac{nN!}{(N-n)!n!} p^n q^{N-n} = \sum_{n=0}^{n=N} p \frac{\partial}{\partial p} \frac{N!}{(N-n)!n!} p^n q^{N-n} \quad (29)$$

Since the variable p is not being summed over, we can bring p and its derivative out of the sum:

$$\begin{aligned} \sum_{n=0}^{n=N} \frac{nN!}{(N-n)!n!} p^n q^{N-n} &= p \frac{\partial}{\partial p} \sum_{n=0}^{n=N} \frac{N!}{(N-n)!n!} p^n q^{N-n} \\ &= p \frac{\partial}{\partial p} (p+q)^N \\ &= pN(p+q)^{N-1} \end{aligned}$$

The equality above is true for any p , q and N . In our case, $p+q=1$ and we have

$$\begin{aligned} \bar{n} &= \sum_{n=0}^{n=N} \frac{nN!}{(N-n)!n!} p^n q^{N-n} \\ &= p \frac{\partial}{\partial p} \sum_{n=0}^{n=N} \frac{N!}{(N-n)!n!} p^n q^{N-n} \\ &= pN(p+q)^{N-1} \\ &= pN \end{aligned}$$

Appendix II: Geiger Counter Time Series

If time permits, we will measure the times between successive recordings of the Geiger Counter. The times appear to be random, and if one were to check for randomness, one would find that they are truly random. We will make a frequency plot

(or histogram) of these times. We can determine the functional form of the frequency plot as follows:

Suppose each radioactive nucleus has a certain probability to decay per unit time. This probability does not depend on how long the nucleus has been in its excited state (i.e. radioactive).

As before, we denote the probability to decay per unit time by the symbol λ , the decay constant of the decay. λ has units of 1/time, and you will see that it only has meaning in the limit as the time goes to zero. The probability that a particular nucleus will decay in the time interval δ is $\lambda\delta$ **in the limit as $\delta \rightarrow 0$** . Our radioactive sample has a large number of radioactive nuclei, N_0 . The probability that one nucleus will decay in the time interval δ is $N_0\lambda\delta$ in the limit as $\delta \rightarrow 0$. If the efficiency of our detector is ϵ , then the probability that our Geiger Counter tube will detect a particle in the time interval δ is $\epsilon N_0\lambda\delta$, in the limit as $\delta \rightarrow 0$. For convenience we define $A \equiv \epsilon N_0\lambda$. Thus, if the decay process is probabilistic, the probability that our Geiger Counter tube will detect a particle in the time interval δ is $A\delta$, in the limit as $\delta \rightarrow 0$.

Now consider our experiment. Since N_0 is very large, it hardly changes during our experiment. So A is essentially the same for every recording. We can ask the following question: What is the probability P_{not} that we will not record a count within a time t since the last count was recorded? This can be answered as follows. Divide the time t into M equal segments, each of duration δ . That is $\delta = t/M$. Then P_{not} is given by

$$P_{not} = (1 - A\delta)^M \quad (30)$$

Now we need to take the limit as $\delta \rightarrow 0$, or as $M \rightarrow \infty$:

$$P_{not} = \lim_{M \rightarrow \infty} \left(1 - \frac{At}{M}\right)^M \quad (31)$$

This limit is the exponential to the base e :

$$P_{not} = e^{-At} \quad (32)$$

In our histogram, we have binned the data in intervals of Δt . The probability that the Geiger counter will receive a signal between the time t and $t + \Delta t$ is therefore:

$$P(t)\Delta t \approx e^{-At} A\Delta t \quad (33)$$

We have used the approximation sign, since A is only defined in the limit as $\delta \rightarrow 0$ and Δt is a finite time. If we collect a total of C_{tot} data points, then the number of counts recorded between time t and $t + \Delta t$ is just

$$C(t) \approx C_{tot} A e^{-At} \Delta t \quad (34)$$

As $C_{tot} \rightarrow \infty$ and $\Delta t \rightarrow 0$ the approximation gets better. If $C(t)$ is an exponential function, then your data are consistent with the probability hypothesis. Maybe God does play dice with the universe?