Lecture 6

The Born Approximation

The Born approximation can be derived from time dependent perturbation theory. The probability for a transition to occur from an initial state $\psi_i \propto e^{ik_i \cdot r}$ to a final state $\psi_f \propto e^{ik_f \cdot r}$ (where $k = p/\hbar$) is proportional to the matrix element

$$< \psi_f|V|\psi_i > \propto \int_0^\infty V(r) e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}} d^3\vec{r} \tag{1}$$

This integral, which is the Fourier transform of V(r) with respect to the momentum transfer, produces a scattering diffraction pattern. The kinematics enters the problem only with the momentum transfer, which is most conveniently defined as $\vec{q} = \vec{k}_f - \vec{k}_i = (\vec{p}_f - \vec{p}_i)/\hbar$.

$$< \psi_f|V|\psi_i > \propto \int_0^\infty V(r) e^{i\vec{q}\cdot\vec{r}} d^3\vec{r} \tag{2}$$

At first glance, the integral looks difficult. However, if we choose the coordinate system for the integral such that the z-axis is directed along $\vec{q}$, then $\vec{q}\cdot\vec{r} = |q|\ |r|\ cos(\alpha) = qr\ cos(\alpha)$, where α is the angle between $\vec{q}$ and $\vec{r}$. With this choice of coordinate system, the integral becomes:

$$\int_0^\infty V(r) e^{i\vec{q}\cdot\vec{r}} d^3\vec{r} = \int_0^\infty V(r) \int_0^\pi e^{iqrcos(\alpha)} 2\pi sin(\alpha)\ d\alpha r^2 dr \tag{3}$$

The angular part of the integral can be evaluated since $sin(\alpha)$ is the derivative of $cos(\alpha)$:

$$\begin{aligned}
\int_0^\infty V(r)e^{iqrcos(\alpha)} 2\pi sin(\alpha) r^2 \, d\alpha dr &= \int_0^\infty V(r) 2\pi (\frac{-e^{iqrcos(\alpha)}}{iqr})|_0^\pi r^2 dr \\
&= \int V(r) 2\pi \, \frac{e^{iqr} - e^{-iqr}}{iqr} \, r^2 dr \\
&= \frac{4\pi}{q} \int_0^\infty V(r) r \; sin(qr) dr
\end{aligned}$$

which is the integral you need to solve numerically.

Note that the angle enters through q, since $q = (2p/\hbar) sin(\theta/2)$. When integrating over r, sin(qr) can change sign and the integral can be zero. That is, there can be angles where this integral will be zero. The cancelation in the integral is in essence destructive interference from the particle scattering off different parts of the target. One sees the same effect in double slit interference. As with the double slit interference pattern, the interference occurs at smaller angles for larger targets. This effect is seen in the comparison of the K^+ scattering data off ^{12}C versus ^{40}Ca. The diffraction interference occurs at around 28° for ^{12}C and around 17° for ^{40}Ca for kaons with the same momentum.

If $V(r)$ is proportional to the density of the target, this integral is often refered to as the **form factor** of the interaction. We note that the Born approximation is an approximation of the complete calculation. An "accurate-numerical" non-relativistic calculation of the cross section can be done by solving the Schroedinger equation with scattering boundary conditions. We will talk about this solution in our next assignment. This assignment introduces you to parameters used in scattering experiments, numerical integration, and relativistic kinematics, which we discuss next.

Relativistic Kinematics

The experimentalists measure the kaon momentum in the laboratory frame. However, it is better to do the analysis (theory) in the center of mass frame. The results we obtain from the Born approximation holds in the center-of-mass reference frame. So for a proper comparison, we need to use the kaon's momentum in the center of mass frame of the kaon and ^{12}C. 800 MeV/c is the kaon's momentum in the lab. I obtain 740 MeV/c for the kaon's momentum in the c.m. frame, and this value should be used in our equation. Let' see how we calculate the c.m. momentum from the lab value.

It might have been a while since you covered relativistic kinematics in your classes, and since high energy interactions is in the relativistic energy and momentum regime, this is a good time to refresh our memory on the topic.

The total relativistic energy as well as the total relativistic momentum for a system of particles are conserved quantities. The relationship between a particle's (say particle "1") energy, E_1, and it's momentum, $\vec{p}_1$ is:

$$E_1^2 = m_1^2 c^4 + p_1^2 c^2 \tag{4}$$

where $p_1^2 = \vec{p}_1 \cdot \vec{p}_1$. For a system that starts with two particles, a and b, and ends with two particles, c and d, the conservation of total energy means:

$$E_a + E_b = E_c + E_d \tag{5}$$

The conservation law holds if a and c, and/or b and d are different particles. Similarly, momentum conservation for two particle interactions means:

$$\vec{p}_a + \vec{p}_b = \vec{p}_c + \vec{p}_d \tag{6}$$

As an example, one can determine the momentum and energy of the pions in the rho meson decay. The rho meson, ρ, can decay into two pions:

$$\rho \rightarrow \pi^+ + \pi^- \tag{7}$$

In the rest frame of the ρ, the pions will travel in opposite direction with equal momentum. This is true, since momentum is conserved. The conservation of energy requires that:

$$\begin{aligned} E_\rho &= E_{\pi^-} + E_{\pi^+} \\ m_\rho c^2 &= \sqrt{m_\pi^2 c^4 + p^2 c^2} + \sqrt{m_\pi^2 c^4 + p^2 c^2} \end{aligned}$$

This equation can be solved for p to give

$$p = c\sqrt{(m_\rho/2)^2 - m_\pi^2} \tag{8}$$

Using $m_\rho = 775\ MeV/c^2$, and $m_\pi = 139\ MeV/c^2$, one obtains $p \approx 362\ MeV/c$.

In addition to conserved quantities, invarient quantities, or invarients, are also important. Invarients are expressions that are the same for all observers. For example, for a particle $E^2 - p^2c^2$ will equal m^2c^4 for any observer. The energy E and the momentum p will depend on one's reference frame, but the combination $E^2 - p^2c^2$ will give the same value in all inertial reference frames.

A useful invarient quantity in particle physics is the invarient total energy, $\sqrt{s}$, of the two particles, since it is the same in all reference frames. For two particles, the invarient total energy squared, s, is given by:

$$s = (E_a + E_b)^2 - (\vec{p}_a + \vec{p}_b)^2 c^2 \tag{9}$$

where the squaring of the vector sum means the scalar product with itself.

To see the usefulness of the quantity s, or $\sqrt{s}$, consider the following example. Suppose you wanted know what the threshold energy is to produce a particle of mass (rest mass) M by scattering a particle of mass m_1 at a target of mass m_2, where m_2 is initially at rest. Note: $M > m_1 + m_2$:

$$m_1 + m_2 \to M \tag{10}$$

The experiment is carried out in the lab reference frame. In this frame the total invariant energy squared is

$$\begin{aligned} s &= (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 c^2 \\ &= (E_1 + m_2 c^2)^2 - p_1^2 c^2 \\ &= E_1^2 + 2E_1 m_2 c^2 + m_2^2 c^4 - p_1^2 c^2 \\ s &= 2E_1 m_2 c^2 + m_1^2 c^4 + m_2^2 c^4 \end{aligned}$$

The total invarient energy is both conserved and invarient. Since it is conserved, its value is the same before the interaction as after. Since it is invarient, we can equate our expression to s in the center of mass frame. Thus, **due to its invarience and conservation, s in the lab frame before the interaction will equal s in the center of mass frame after the interaction**. After the interaction, there is only one particle of mass M and it is at rest in the center of mass frame. The particle M will be at rest, since we are interested in the threshold energy E_1 to produce the particle. Hence, $s = M^2 c^4$ after the collision:

$$\begin{aligned} 2E_1 m_2 c^2 + m_1^2 c^4 + m_2^2 c^4 &= M^2 c^4 \\ &\textit{or} \\ E_1 &= \frac{M^2 - m_1^2 - m_2^2}{2m_2} c^2 \end{aligned}$$

Suppose we want to produce a delta particle, mass $M = m_\Delta = 1232\ MeV/c^2$, by scattering a pion, mass $m_\pi = 139\ MeV/c^2$, off a proton, mass $m_p = 940\ MeV/c^2$. Then, to produce the $\Delta(1232)$ particle the pion needs to have a total energy of

$$E_\pi = \frac{1232^2 - 139^2 - 940^2}{2(940)} \approx 327 MeV \tag{11}$$

or a kinetic energy in the lab frame of $K.E. \approx 327 - 139 = 188 MeV$. If the pion has an energy less than 327 MeV, then the Δ particle cannot be produced. In this case, most likely the pion would scatter elastically off the proton. If the pion has an energy a little greater than 327 MeV, then the Δ could be formed, which would decay back into a pion plus proton. If the pion had enough energy, then a Δ plus another particle (a pion) could be the final particles produced.

Now let's consider the kinematics involved in our assignment. We know the laboratory momentum of the kaon, $p_{lab} = 800\ MeV/c$. We need to know the momentum of the kaon in the center of mass reference frame of the kaon and the nucleus (^{12}C in our case). The energy of the kaon in the lab frame is

$$E_K = \sqrt{m_K^2 c^4 + p_{lab}^2 c^2} \tag{12}$$

It will be convenient to work with the total invarient energy. From the previous calculation we know

$$s = m_K^2 c^4 + m_N^2 c^4 + 2 E_K m_N c^2 \tag{13}$$

where m_N is the mass of the nucleus. Since s is an invarient, it's value is the same in all inertial reference frames. Thus, we can equate s in the lab frame to s in the center of mass frame. This equality will allow us to calculate the kaon's momentum in the center of mass frame.

For two particle systems, the two particles will have equal and opposite momenta in the center-of-mass reference frame, $\vec{p}_1 = -\vec{p}_2$. Letting $|\vec{p}_1| \equiv p$, we have

$$s = (\sqrt{m_1^2c^4 + p^2c^2} + \sqrt{m_2^2c^4 + p^2c^2})^2 - 0^2 \tag{14}$$

The last term is zero, since $\vec{p}_1 + \vec{p}_2 = 0$. After a bit of algebra, one can solve for p in the center of mass frame

$$p^2 = \frac{(s - (m_1c^2 + m_2c^2)^2)(s - (m_1c^2 - m_2c^2)^2)}{4sc^2} \tag{15}$$

in terms of the total invarient energy squared, s.

With these equations, we can calculate the kaon's momentum in the center of mass frame in terms of its lab frame momenta, p_{lab}. First we determine E_K in the lab frame:

$$E_K = \sqrt{m_K^2c^4 + p_{lab}^2c^2} \tag{16}$$

Then we calculate s in the lab frame.

$$s = m_K^2c^4 + m_N^2c^4 + 2E_Km_Nc^2 \tag{17}$$

Finally, we solve for p in the center-of-mass frame using

$$p^2 = \frac{(s - (m_Kc^2 + m_Nc^2)^2)(s - (m_Kc^2 - m_Nc^2)^2)}{4sc^2} \tag{18}$$

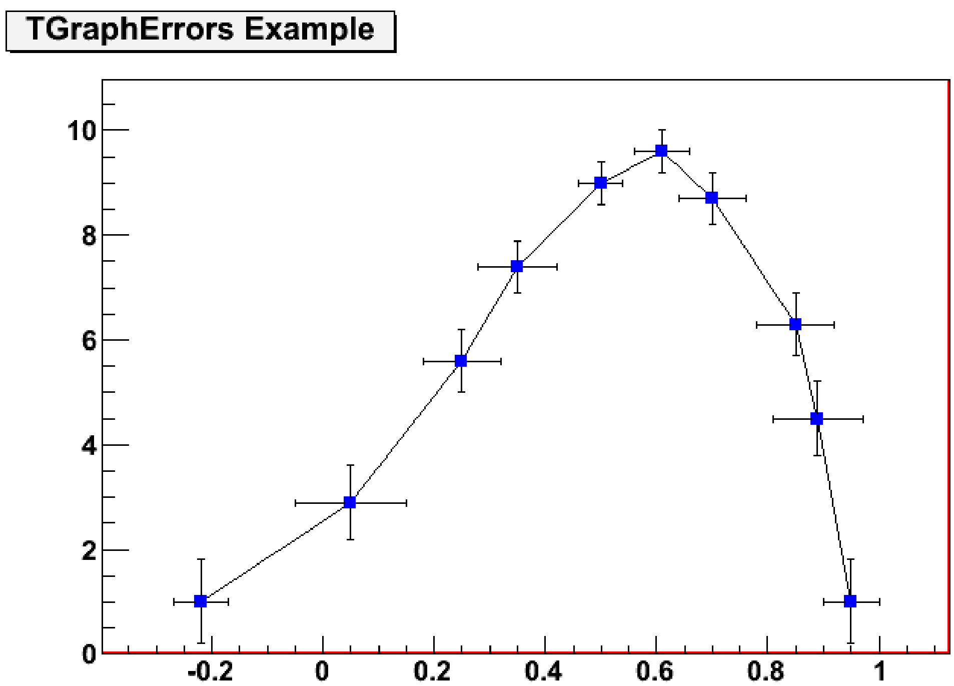
For $m_K = 493\ MeV/c^2$, $m_N = 12(940)\ MeV/c^2$, and $p_{lab} = 800\ MeV/c$, I obtain $p = 740\ MeV/c$. I didn't use my calculator, I just had the computer do the work with the following code in my homework 3 code:

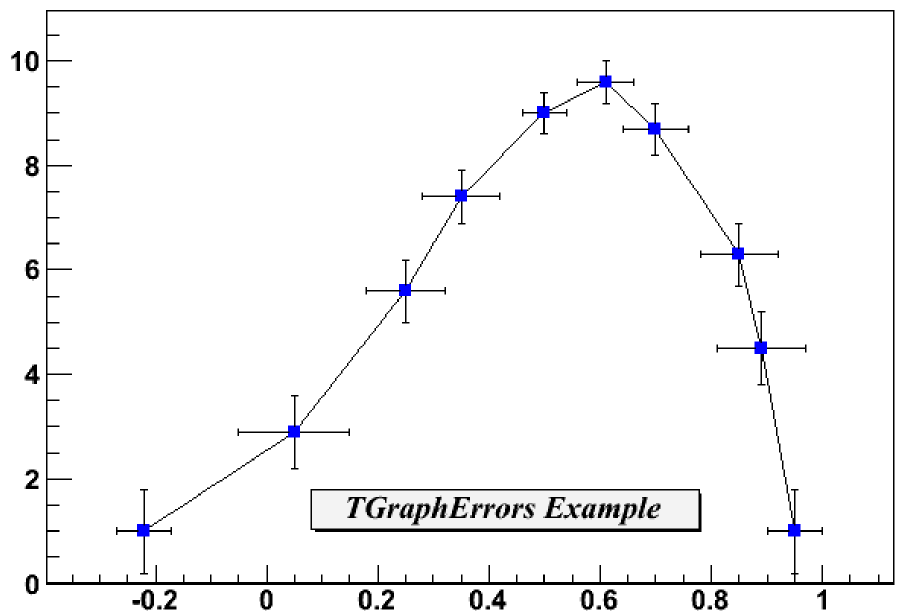
```
mkaon=493.0;
mnuc=12.0*940.0;
plab=800.0;
ekaon=sqrt(mkaon*mkaon+plab*plab);
s=mkaon*mkaon+mnuc*mnuc+2.0*ekaon*mnuc;
pc2=(s-(mkaon+mnuc)*(mkaon+mnuc))*(s-(mkaon-mnuc)*(mkaon-mnuc))/4.0/s;
pc=sqrt(pc2);
```

Graphing in Root

Below is a code from the CERN ROOT tutorials. It is a simple code which can be used to graph one data set:

```
void Ggraph()

//Draw a graph with error bars

TCanvas *c1 = new TCanvas("c1","A Simple Graph with error bars",200,10,700,500);
c1→SetFillColor(0);
// c1→SetGrid();
c1→GetFrame()→SetFillColor(0);
c1→GetFrame()→SetBorderSize(0);

const Int_t n = 10;
Float_t x[n] = -0.22, 0.05, 0.25, 0.35, 0.5, 0.61,0.7,0.85,0.89,0.95;
Float_t y[n] = 1,2.9,5.6,7.4,9,9.6,8.7,6.3,4.5,1;
Float_t ex[n] = .05,.1,.07,.07,.04,.05,.06,.07,.08,.05;
Float_t ey[n] = .8,.7,.6,.5,.4,.4,.5,.6,.7,.8;

TGraphErrors *gr = new TGraphErrors(n,x,y,ex,ey);
gr→SetTitle("TGraphErrors Example");
gr→SetMarkerColor(4);
gr→SetMarkerStyle(21);
gr→Draw("ALP");

c1→Update();
```


TGraphErrors Example
10
8
6
4
2
0
-0.2
0
0.2
0.4
0.6
0.8
1


TGraphErrors Example
10
8
6
4
2
0
-0.2
0
0.2
0.4
0.6
0.8
1