

Appendix A: Numbers

Scientific Notation

Any number may be expressed in “scientific notation” by factoring out whatever power of ten is necessary to leave a value that is greater than or equal to 1 and less than 10. For example,

$$10 \Rightarrow 1.0 \times 10^1 \quad 753.24 \Rightarrow 7.5324 \times 10^2 \quad 0.0000354 \Rightarrow 3.54 \times 10^{-5}$$

Significant Digits

When the uncertainty in a value is *not* explicitly reported (as we will discuss in a later section) one can still form an impression of how accurately the value is known from the number of “significant digits” (sometimes called “significant *figures*” or “sig figs”) used to report it. The number of significant digits in a value is determined as follows:

- 1 Find the “most significant digit” (MSD). This is the first *non-zero* digit that you come to reading from left to right.
(For example “4” is the MSD in both 432.7 and 0.0004971.)
- 2 Find the “least significant digit” (LSD). This is the last digit that you come to reading left to right that is *not* a “place-holder.”
(For example “7” is the least significant digit in both 5900.7 and 0.0347 and the final “0” is the least significant digit in both 233.0 and 0.008550. On the other hand it is not immediately evident which is the least significant digit in 34,500. Using scientific notation resolves the ambiguity. If the number had been written as 3.450×10^4 we would know that the zero immediately following the five is the last significant digit and the next zero is merely a place-holder.)
- 3 To find the number of significant digits, start with the MSD and count *every* digit up to and including the LSD.
(For example, there are four significant digits in 300.5, 0.007331, and 5.220×10^{12} ; there are two significant digits in 34, 1.0, and 7.3×10^{-6} ; and there is one significant digit in 4, 0.00005, and 8×10^7 .)

There are several rules for determining the appropriate number of significant digits to report in the result of a calculation:

- 1 When adding and/or subtracting numbers: Determine the number that has its LSD in the highest decimal *place*. Round the result so that its LSD occupies this same decimal place.

If, for instance, we know nothing about the digit in the “tens place” of one number in a sum or difference, we cannot *possibly* know about the digit occupying the “tens,” “ones,” or any smaller decimal place in the result.

As an example, let’s perform the calculation

$$1.20 \times 10^5 + 27.7654133 - 4200$$

It will help to express each number with a common power of ten

1.20	$\times 10^5$	
+0.000277654133	$\times 10^5$	
-0.042(00?)	$\times 10^5$	
1.15827.7654133	$\times 10^5$	(without regard for sig figs)
1.16	$\times 10^5$	(properly rounded)

Notice that, even though the zeroes in 4200 have questionable significance, they are both less significant than the zero in 1.20×10^5 . Also notice that, even though the second number has nine significant digits, it does not contribute to the result at all!

2 When multiplying and/or dividing numbers: Determine the individual number that has the smallest *number* of significant digits. Round the result to the same *number* of significant digits.

This rule is a result of the fact that the number of significant digits gives a rough idea of the *relative* accuracy of a number. When multiplying and dividing numbers, the result has roughly the same *relative* accuracy as that of the least accurate individual number.

As an example check for yourself that

$$= 8.312529 \times 10^{-4} \quad (\text{without regard for sig figs})$$

$$= 8.3 \times 10^{-4} \quad (\text{properly rounded})$$

The number of significant digits in the result is determined from the fact that 5.6×10^3 has the smallest number of significant digits—i.e., two.)

3 When performing a series of calculations, retain one or two extra significant digits and round appropriately at the *end* of the calculation. This will prevent the accumulation of “round off errors.”

4 Integers and values like “ π ” that appear in formulas are considered to have an infinite number of significant digits. Just make sure that you actually *use* a sufficient number of significant digits to guard against round off error.

Exercises:

Express the following in scientific notation with the correct number of significant digits:

a) 34,254.33 b) 0.0025 c) 71.30 d) 265×10^5 e) 0.00400

f) $73.1 + 25.3 - 56.2 + 33.7 + 17$ g) $234 + 3.11 + 73.2$ h) $34.5 \times 24 \times 0.02275$

i) $4560 + 3.5 \times 10^2 - 43$ j) k) $(2.3 \times 10^{-3})(5.880 \times 10^4) + 8.44$

Appendix B: Units

Pure numbers are good for counting, but most quantities of interest in science have units (e.g., “kg,” “m,” “m/s,” etc.) The value of such a quantity cannot be understood without the units. It is meaningless to say, “The length of the table is 5,” or, “It took me 3 to finish my homework.” Admittedly, statements like, “I weigh 160,” and, “That car was doing 75,” will be understood by most people because—at least in the United States—they will automatically supply the missing units, “pounds” and “mph,” respectively. In reporting data, however, you must *always* specify the units you are using.

Units are themselves *mathematical* entities that obey the same algebraic rules that numbers do. While it is true that “100 cm = 1 m,” it is certainly *not* true that “100 = 1” or that “cm = m.” Think of “100 cm = 1 m” as *literally* meaning “100 times centimeter equals 1 times meter.” Therefore, we can also write “= 100” or “= 1.”

Often, after performing a calculation, you will find that the answer is expressed in unconventional and/or inconvenient units. Consider the following calculation:

$$t = \sqrt{\frac{4.57 \text{ κμ}}{980 \text{ χμ/σ}^2}} = \sqrt{4.663 \xi 10^{-3} \frac{\text{κμ σ}^2}{\text{χμ}}} = 6.83 \xi 10^{-2} \sigma \sqrt{\frac{\text{κμ}}{\text{χμ}}}$$

Since we have properly operated on the units as algebraic entities, this is a *completely* valid answer for the time t , but it *is* expressed in rather unconventional units.

Converting Units: The “Multiply by One in the Form of” Method

What if we want to express the time calculated above in more conventional units like “seconds”? Converting units is *always* easily accomplished using the “multiply by one in the form of” method. We need only remember that “1 km = 1000 m” and “1 m = 100 cm.” Therefore,

$$1 \qquad \qquad \text{and} \qquad \qquad 1$$

Of course, we can raise 1 to any power and still have 1; so, for instance

$$1 \qquad \qquad \text{and} \qquad \qquad 1$$

Now we simply write

$$6.83 \times 10^{-2} \text{ s} \sqrt{\frac{\text{km}}{\text{cm}}} = 6.83 \xi 10^{-2} \sigma \sqrt{\frac{\text{κμ}}{\text{χμ}}} \cdot \sqrt{\frac{1000 \text{ μ}}{\text{κμ}}} \cdot \sqrt{\frac{100 \text{ χμ}}{\text{μ}}} = 21.6 \sigma$$

Please note the following:

- We have multiplied the right side of the equation by two factors of 1 so we have *not* changed the *value* of the quantity; we have only *converted* the *units* in which it is expressed. This is why our “one in the form of” factors are called “unit conversion factors.”
- We have chosen the particular conversion factors based on the conversion we wish to accomplish. In general, insert conversion factors that will algebraically cancel the units you are trying to convert.

As a final example, suppose that we actually wanted to express the time t in hours. Since “1 min = 60 s” and “1 hr = 60 min,” we would write

$$21.6 \text{ s} = 21.6 \sigma \cdot \frac{1 \mu\nu}{60 \sigma} \cdot \frac{1 \eta\rho}{60 \mu\nu} = 6.00 \xi 10^{-3} \eta\rho$$

Exercises:

1. Use the “multiply by one in the form of” method to express...

- | | | | |
|--------------|--------|--------------------------|----------|
| a) 79.3 cm | in m. | f) 35.6×10^{-2} | in N. |
| b) 0.0450 kg | in g. | | |
| c) 3.22 min | in s. | g) 3.44×10^3 | in . |
| d) 49 inches | in cm. | | |
| e) 23 | in . | h) 2.77 | in days. |

2. Why is it *impossible* to “convert” pounds to kg? ...m² to liters? ...etc.

Appendix C: Analysis of Uncertainty

In an experiment, we make direct measurements of quantities like lengths, times, masses, etc. The instruments used to make these measurements vary in quality as does the expertise of the researchers using them. Furthermore, parameters that are out of the control of the researcher may cause undesired variations in the things that are being measured. Often, the very thing being measured may not even *have* a well-defined value like, for example, the diameter of a cylinder that isn't perfectly round (as is *always* the case to some degree).

These measured values are used in calculations to obtain experimental results that are usually compared with the predictions of theory or with the results of other experiments. Since experimental results are based on measurements that are *unavoidably* uncertain, there will inevitably be *some* difference between the values being compared. Thus, the important question arises, “Is the difference more than could reasonably be expected *given* the uncertainties in the measured values?” Put more simply we mean, “How likely is it that the difference is *real*?”

“Analysis of uncertainty” (often misleadingly called “*error* analysis”) is a three step process that consists of 1) estimating the unavoidable uncertainty in one's experimental measurements, 2) determining the resulting uncertainty in the calculated results, and 3) using those uncertainties to evaluate the significance of experimental discrepancies.

As you read the following sections and as you perform the experiments in this course please keep these points in mind:

- 1 Estimates of uncertainty are just that—estimates. For our purposes they are *intended* to represent “worst case” values—i.e., your estimate of the largest amount by which the *true* value of a quantity might differ from the *measured* value. Although estimates of uncertainties are never “guesses,” reasonable people may assign uncertainties to the same measurement that vary by a factor of two or three. Keeping this in mind can significantly simplify the analysis of uncertainty and improve your ability to evaluate your results.
- 2 *Genuine* discrepancies between experiment and theory *do* occur even in student labs. These are generally due to *real* effects that have not been taken into account in the theory used to analyze the data. One of the most interesting and challenging aspects of data analysis is identifying possible reasons for such genuine discrepancies.
- 3 Notwithstanding the above, if you are reasonably careful in the lab, you will *rarely* observe discrepancies larger than 20 or 30%. Whenever you *do* obtain discrepancies that large it is *quite* likely that you have made a mistake either in collecting your data or in performing the required calculations. Please recognize the *enormous* difference between a discrepancy of 10% and one of a *factor* of ten. The former may arise as a result of unavoidable uncertainties in the data; the latter is almost *certainly* the result of a mistake.

We will use the following notation and terminology in all that follows:

Let “ x ” be the name of a measured or calculated quantity. Then

x_{true} = the “*true value* of ‘ x .’” (Usually unknown and unknowable!)

x_{best} = our “*best estimate* of x_{true} ”

δx = the “*absolute uncertainty* in x_{best} ,” i.e., the largest likely value of $|x_{\text{best}} - x_{\text{true}}|$. (Often called simply *the uncertainty*.)

$\frac{\delta x}{x_{\text{best}}}$ = the “*relative or fractional uncertainty* in x_{best} ,” often expressed as a percentage and then called the “*percentage uncertainty* in x_{best} .”

x_{min} = $x_{\text{best}} - \delta x$ = the *minimum* likely value of x_{true} .

x_{max} = $x_{\text{best}} + \delta x$ = the *maximum* likely value of x_{true} .

x \equiv x_{min} to x_{max} = $x_{\text{best}} \pm \delta x$ = our “*experimental range* for ‘ x ’” (Usually just called the “*experimental value* of ‘ x .’”)

Uncertainties in Measurements

Estimating the uncertainty in measurements is the first step in the analysis of uncertainties and it *must* be done *at the time that the measurements are made*; it can not be done correctly as an afterthought.

Sources of uncertainty may be grouped into two categories

Uncertainty due to instrumental precision: For our purposes the instrumental precision can be determined simply from the scale of the instrument and how accurately you can read it. (More properly it also includes random effects that occur in the *operation* of the instrument and also some understanding of how well the instrument is *calibrated*.) Instrumental precision sets a *lower limit* on the uncertainty in a measurement. Please note carefully, however, that it is *quite* rare for the uncertainty in a measurement to be determined by this factor. One of the most common mistakes made by students in the introductory lab is *underestimating* the uncertainty in a measurement by assuming that it is equal to the instrumental precision and ignoring the generally larger “sample uncertainty.”

Sample uncertainty: Sample uncertainty is determined by the “inherent variability” in the measurement being made. For instance, if one is timing the fall of a ball with a digital stopwatch the sample uncertainty is likely to be *larger* than the instrumental precision because of the difficulty of insuring that it is always dropped from the same height, that it isn’t given a slight push, that you start and stop the stopwatch at the right times, etc. Measuring the width of a table with a meter stick would probably have less sample uncertainty unless, of course, the width actually *does* vary along its length. Sample uncertainty also results from the role of the researcher’s judgment in using the instrument.

It is worth taking a moment to identify two types of *errors* that may afflict experimental measurements. We have *neither* of these in mind when we discuss experimental *uncertainty*.

Mistakes, blunders, etc.: These come in *many* forms and include such lapses of technique as misreading instruments, forgetting to zero an instrument, making calculation errors, mistakes in reporting units, etc. While you may occasionally make mistakes, they are *not* legitimate sources of *uncertainty*. They should be discovered and eliminated. With proper care you won't make them.

Systematic errors: Influences that produce consistent deviations in a set of measurements. For instance, they may result from using uncalibrated instruments. These are usually the hardest kind of errors to discover. We will say more about systematic errors when we discuss the evaluation of experimental discrepancies.

Example: Finding the mass of an object

Suppose you are measuring the mass “*m*” of an object on a triple-beam balance as shown in Fig. 1. First you make certain that the balance is “zeroed”—i.e., that the indicator points to zero when all the balance weights are set to their “zero” positions and there is no mass in the pan. (Even this preliminary procedure is part of the measurement process; it involves your judgment; and it is a source of uncertainty in the measurement.) Next you place the object to be measured in the pan and adjust the two larger calibrated masses on the measuring unit as shown in Fig. 1 to be near the balance condition. The two larger calibrated masses rest in indents assuring you that they are placed properly. (How *well* do they “assure you” of that? Would they be in *exactly* the same position if you were to make the measurement again?) Finally you adjust the smallest of the three calibrated masses until the pointer on the unit indicates that it is in balance. (Notice carefully that this calls, again, for a judgment on your part. You could *certainly* move the small mass by a bit and still *consider* the scale to be “balanced.”)

Figure 1: The measuring unit of a triple-beam balance

The best way to determine the uncertainty in a measurement is to make the measurement—starting from scratch—many times. Each *individual* measurement will be subject to the limitations imposed by the instrumental precision. By making a set of measurements, however, we will see if there is additional sample uncertainty over and above the instrumental uncertainty. As a result we will form a good idea about *both* the best value (m_{best}) *and* its uncertainty (δm). For instance, suppose that you measured the mass of the same object six times with the balance beam and got the following values:

23.44 g	23.50 g	(Six single measurements of the mass, each subject to the same instrumental precision of about ± 0.01 g.)
23.31 g	23.35 g	
23.47 g	23.38 g	

Note the following:

- The values are all between 23.31 g and 23.50 g.
- Their average is 23.4083 g.
- They range fairly uniformly above and below this value by about 0.1 g.
- No values are particularly out of line with the rest as, say, 25.10 g would be.

A reasonable conclusion is that the measurements are “consistent” with each other and simply reflect the sample uncertainty that results from our judgment in the use of the scale. The measurements imply an uncertainty δm of about 0.1 g. Given this, it makes no sense to report our best value with digits beyond the tenth’s place. Accordingly we might take 23.4 g for m_{best} and 0.1 g for δm and report this concisely as

$$m \equiv m_{\text{best}} \pm \delta m = (23.4 \pm 0.1) \text{ g} = (2.34 \pm 0.01) \times 10^{-2} \text{ kg}$$

Another reasonable person might decide that the “true value” probably lies somewhat closer to the average than to the outlying values and might report the result as

$$m = (23.41 \pm 0.05) \text{ g} \quad (\text{Another } \textit{reasonable} \text{ way to report the result})$$

In any event observe the following rules:

Rules for reporting experimental values

- 1 Get a *reasonable* value for the uncertainty.
- 2 Round the *uncertainty* to *one significant digit*. (One possible exception to this rule: *If* that digit is a “1” or a “2” you *may* want to keep one *extra* digit. Rounding 0.14 to 0.1 is a 30% change; rounding 0.84 to 0.8 is only a 5% change!)
- 3 Round the *best value* to the *same decimal place* as the *single* digit in the uncertainty. (*If* you have kept two digits in the uncertainty, then round the best value to the same decimal place as the *second* digit in the uncertainty.)
- 4 Report the best value and its uncertainty with units and any common powers of ten factored out as shown in the examples above.

Nobody, can tell you the “right values” to report for the best value and its uncertainty; these involve your judgment. But, while there is no “right answer,” there are *definitely* wrong answers. In the current case, it would *definitely* be wrong to say any of the following

$$m = 23.4083 \quad (\text{Wrong for lots of reasons, most notably because it doesn't include units.})$$

$$m = 23.4083 \text{ g} \quad (\text{O.K., it has units, but it's } \textit{still} \text{ wrong because it doesn't give the uncertainty.})$$

$$m = (23.4083 \pm 0.0001) \text{ g} \quad (\text{O.K., now we have an uncertainty, but it's } \textit{much} \text{ too small; any } \textit{reasonable} \text{ person would agree.})$$

$$m = (23.4083 \pm 0.0736) \text{ g} \quad (\text{A } \textit{reasonable} \text{ uncertainty reported with too many digits; } \textit{uncertainties} \text{ themselves are seldom, if ever, known that accurately.})$$

$$m = (23.4083 \pm 0.07) \text{ g} \quad (\text{A reasonably reported uncertainty, but the best value reports digits that are } \textit{not} \text{ significant.})$$

$$m = (23 \pm 1) \text{ g} \quad (\text{Correct agreement between the uncertainty and the best value, but the uncertainty is simply not that big; any } \textit{reasonable} \text{ person would agree.})$$

Uncertainties in Calculations: “Propagation of Uncertainty”

Once we have our measured values and reasonable estimates of the uncertainties in them, we are ready to do the calculations that give us our experimental results. We need to obtain *reasonable* estimates for the *uncertainty* in those results. The process by which this is done is called “propagation of uncertainty.” The following “Brute Force” method determines the uncertainty in a result in a rather direct manner.

The “Brute Force” method for propagating uncertainties

Consider a quantity f that is to be calculated from measurements. Calculate f_{\max} and f_{\min} based on appropriate combinations of the maximum and minimum values for the measured quantities. From these determine f_{best} and δf as follows:

$$f_{\text{best}} = \frac{1}{2} (f_{\max} + f_{\min}) \text{ and } \delta f = \frac{1}{2} (f_{\max} - f_{\min})$$

Although the Brute Force method *sounds* straightforward, it is *not* always easy to carry out in complex calculations. Therefore, we will also develop a set of three simple rules that can be used to propagate uncertainties through complex calculations one step at a time.

Uncertainties in Addition and Subtraction

Suppose that we want to add the two measured lengths

$$L_1 = (25 \pm 1) \text{ cm} \quad \text{and} \quad L_2 = (20 \pm 2) \text{ cm}$$

to get $L = L_1 + L_2$. Remember that “ 25 ± 1 ” means “somewhere between 24 and 26” and “ 20 ± 2 ” means “somewhere between 18 and 22.” The Brute Force method would have us find L_{\max} and L_{\min} like this:

$$L_{\max} = L_{1\max} + L_{2\max} = 26 \text{ cm} + 22 \text{ cm} = 48 \text{ cm}$$

and

$$L_{\min} = L_{1\min} + L_{2\min} = 24 \text{ cm} + 18 \text{ cm} = 42 \text{ cm}$$

In other words the range for L is between 42 cm and 48 cm. Accordingly, we would write

$$L = (45 \pm 3) \text{ cm}$$

Note, however, that there is a simpler method since it is no accident that $\delta L = \delta L_1 + \delta L_2$. It is easy to show, in general, that when *adding or subtracting* numbers, the *absolute* uncertainties *add*. This is the first of our three rules for propagating uncertainties through calculations.

Uncertainties in Multiplication and Division

Next suppose that we want to divide the quantity

$$V = (100 \pm 3) \text{ cm}^3 \quad \text{by} \quad A = (20 \pm 1) \text{ cm}^2$$

to get $L = V/A$. It should be clear that

$$L_{\max} = V_{\max}/A_{\min} = (103 \text{ cm}^3)/(19 \text{ cm}^2) = 5.42 \text{ cm}$$

and

$$L_{\min} = V_{\min}/A_{\max} = (97 \text{ cm}^3)/(21 \text{ cm}^2) = 4.62 \text{ cm}$$

So we would get $L = (5.02 \pm 0.40) \text{ cm}$ which, after rounding appropriately, becomes

$$L = (5.0 \pm 0.4) \text{ cm}$$

Once again, however, there is a simpler method. Note that

$$\frac{\delta L}{L_{\text{best}}} = \frac{dV}{V_{\text{best}}} + \frac{dA}{A_{\text{best}}}$$

i.e., $\quad = + \text{ or } \quad 8\% = 3\% + 5\%$

Is this an accident? In fact, no. It is relatively easy to show, in general, that when *multiplying* or *dividing* numbers, the *relative* uncertainties *add*. This is the second of our three rules.

Uncertainties in Powers

Finally, suppose that we want to take the square root of

$$A = (25 \pm 1) \text{ cm}^2$$

to get $L = A^{1/2}$. It should be clear from the Brute Force method that

$$L_{\max} = A_{\max}^{1/2} = (26 \text{ cm})^{1/2} = 5.099 \text{ cm}$$

and

$$L_{\min} = A_{\min}^{1/2} = (24 \text{ cm})^{1/2} = 4.899 \text{ cm}$$

So we would get $L = (4.999 \pm 0.100) \text{ cm}$ which, after rounding appropriately, becomes

$$L = (5.0 \pm 0.1) \text{ cm}$$

Once again, however, there is a simpler method. Note that

$$\frac{\delta L}{L_{\text{best}}} = \frac{1}{2} \frac{dA}{A_{\text{best}}}$$

i.e., $\quad = \text{ or } \quad 2\% = 4\%$

Is *this* an accident? Again, no. It is relatively easy to show, in general, that when raising a number to a power, its *relative* uncertainty is multiplied by the power. This is the last of our three rules which we now summarize as the “step-by-step” method.

Step-by-Step method for propagating uncertainties*Rule #1—Addition and Subtraction*

When adding and/or subtracting a set of numbers, the *absolute* uncertainty in the result is equal to the *sum* of the *absolute* uncertainties in the set of numbers, i.e.,

$$\text{If} \quad f = w + x + \dots - y - z - \dots$$

$$\text{Then} \quad \delta f = \delta w + \delta x + \dots + \delta y + \delta z + \dots$$

(Note: The uncertainties are *all added*, never subtracted.)

Rule #2—Multiplication and Division

When multiplying and/or dividing a set of numbers, the *relative* uncertainty in the result is equal to the *sum* of the *relative* uncertainties in the set of numbers, i.e.,

$$\text{If} \quad f = \frac{\omega \xi \mathbf{K}}{\psi \zeta \mathbf{K}}$$

$$\text{Then} \quad \frac{\delta f}{f_{\text{best}}} = \frac{dw}{w_{\text{best}}} + \frac{dx}{x_{\text{best}}} + \mathbf{K} + \frac{dy}{y_{\text{best}}} + \frac{dz}{z_{\text{best}}} + \mathbf{K}$$

(Note: The relative uncertainties are *all added*, never subtracted.)

Rule #3—Powers

When raising a number to a power, the *relative* uncertainty in the result is equal to the *relative* uncertainty in the original number *times* the power, i.e.,

$$\text{If} \quad f = x^n$$

$$\text{Then} \quad \frac{\delta f}{f_{\text{best}}} = n \frac{dx}{x_{\text{best}}}$$

An Example

Suppose that

$$r = (a - b/c)^2$$

where $a = (12.3 \pm 0.1)$ cm, $b = (72 \pm 2)$ cm³, and $c = (8.5 \pm 0.3)$ cm². Since the first thing we are going to do is a division, we need the *relative* uncertainties in b and c so we can add them. (Rule #2) We will *express* the relative uncertainties as percentages. For instance, the relative uncertainty in b is $= .028 = 2.8\%$.

$$\begin{aligned} r &= \\ &= \\ &= \end{aligned}$$

(While we are doing the calculation we will keep “too many” digits in our numbers to avoid accumulating round off error. We will round off properly at the *end* of the entire calculation.)

Next we will do a subtraction. We need the *absolute* uncertainty in the last term so that we can add it to that of the first term. (Rule #1) We find that 6.3% of 8.47 cm is 0.53 cm so

$$\begin{aligned} r &= \\ &= [(3.83 \pm 0.63) \text{ cm}]^2 \end{aligned}$$

Next, we are going to square this value. We need to go *back* to the relative uncertainty again so that we can double it. (Rule #3)

$$\begin{aligned} r &= [3.83 \text{ cm} \pm 16.4\%]^2 \\ &= 14.7 \text{ cm}^2 \pm 33 \% \end{aligned}$$

Finally, we express the result in standard form, rounding appropriately.

$$r = (15 \pm 5) \text{ cm}^2$$

Just to check, let's try the Brute Force method

$$r_{\min} = [(12.2 - 74/8.2)\text{cm}]^2 = 10.1 \text{ cm}^2$$

$$r_{\max} = [(12.4 - 70/8.8)\text{cm}]^2 = 19.8 \text{ cm}^2$$

So we get $r = (14.95 \pm 4.85) \text{ cm}^2$ which, after rounding appropriately, becomes

$$r = (15 \pm 5) \text{ cm}^2$$

In agreement with the result from the Step-by-Step method

Evaluating the “Significance” of “Discrepancies”

Usually at the end of an experiment we will compare our results with those of others or the predictions of some theoretical calculation. This will involve comparing two or more values to see how well they agree with each other. Any difference between two such values is called a “discrepancy.” Keep in mind, however, that discrepancies are *inevitable* since the results of experimental work will always be subject to experimental uncertainty. The important question is, “*Is the discrepancy within the limits set by the experimental uncertainty?*” If it is, we say the discrepancy “is *not* significant;” that is, it is *accounted for* by the size of the experimental uncertainty. If the discrepancy is outside the limits set by the experimental uncertainty, we say the discrepancy “*is* significant;” that is, it is *not* accounted for by the size of the experimental uncertainty. (Please keep in mind that this is never an “all or nothing” judgment; “significant” discrepancies vary from “somewhat significant” to “extremely significant.”)

When the discrepancy is judged to be “significant” we have some more work to do. Without question, the *first* thing we must do is to check our calculations and make sure that we haven’t simply made a mistake.

If our calculations are correct we should consider the possibility that we have underestimated the experimental uncertainties. Generally this can only be done by taking more data and reconsidering our estimates of the uncertainty in our measurements. If we find that we have underestimated the uncertainties, a reanalysis of the data may render the discrepancy “insignificant.”

If we *still* find that the discrepancy is significant, we would want to take a look at the numbers with which we are comparing our results—especially if they are the result of theoretical calculations. The predictions of theory are only as reliable as the theory upon which they are based. For example, the predicted range R of a projectile shot horizontally with initial speed v off of a cliff with height h is

$$R = v \sqrt{\frac{2h}{g}} \quad (\text{where } g = \text{the acceleration due to gravity})$$

This prediction assumes, however, that 1) there is *no* air resistance, 2) the projectile is shot *exactly* horizontally, 3) h and R are measured between appropriate points—not a trivial task, 4) the acceleration due to gravity is *constant*, etc. Some of these assumptions are likely to be *very* good, but others may be highly questionable. Ignoring these factors introduces systematic errors into the experiment which, in turn, will produce non-random discrepancies in the results. When looking for this type of error pay attention to the algebraic sign of the observed discrepancy. In the case of the projectile, for instance, if we observe that our *experimental* range is consistently *larger* than the range predicted from *theory*, we can rule out air resistance as a possible explanation.

If we still judge the discrepancy significant after careful consideration of *every* thing that might *possibly* account for it *and if* we can convince others to do the same experiments, reproduce our discrepancies, and agree that they are significant (a *big* if!), then it *may* be time to celebrate. For the discovery of a *genuinely* significant discrepancy is an important event in an *experimental* science like physics. No matter how deeply we believe in a theory, in spite of how many other phenomena it explains, and, certainly, regardless of how many Nobel Prizes its developers may have won, if it *really* does not agree with the results of experiment, it is wrong.

Appendix D: Data Tables

Good data tables are designed to clearly communicate values obtained from a set of similar measurements. Please look at the example on the next page and note the following items:

- 1 Tables are numbered and titled meaningfully.
- 2 Constant values are listed under the title. *Identify* the quantities with a few words; display the *symbols* that you will use consistently throughout the report to represent the quantities; and, finally, give the *values* of the quantities along with their *units* and *uncertainties*. The values of *all* quantities needed to evaluate the experiment must be listed either here or in the table.
- 3 The table itself is a system of rows and columns containing the data that is obtained from individual measurements and headers that clearly indicate what those data represent. The specific organization may vary depending upon the nature of the experimental data. Most often, however, the *columns* will represent the various quantities that are associated with the trial runs and each *row* in the body will represent the data from a *single* trial run. If this is the case, columns should be organized starting with the variable parameters that you *choose*, followed by the data that you directly *measure*, the values that you *calculate*, and culminating in those that you *graph* if a graph is to be made from the data. Rows should be organized in increasing or decreasing order of some fundamentally important parameter, even if the data were not originally *taken* in that order.
- 4 Headers are used to describe the data and to eliminate as much repetitive information as possible from the body of the table. They will include 1) a few words to *identify* the data in the body of its column (or row), 2) the *symbol* used to represent that data, 3) the *power of ten* and the *units* to be attached to the entries in the body, and 4) the *uncertainty* to be attached to the entries (only if those uncertainties can be expressed as a constant).
- 5 The body contains the individual values. Each entry in the body contains the minimum information necessary so that relationships between observations and trends in the data stand out. Redundant information is included in the headers and each number is rounded to an appropriate number of digits. If necessary a common power of ten should be factored out and included in the header (as already noted in item 4) so that individual entries remain *reasonably* close to 1—perhaps between 0.1 and 100.
- 6 Some items may require explanatory notes. These are keyed to the relevant items and appear below the table.

Making complete, compact, and easily read tables is an art. Different people may choose different but equally good ways to represent the same data. The important point, as in *all* written communication, is that you consider the needs of your audience and make *their* job as easy as possible.

Sample Data Table

(Title) Table 1: Atwood machine data with total mass held constant

(Constants)

Mass of cup #1	M_{1c}	$= (27.5 \pm 0.1) \times 10^{-3} \text{ kg}$
Mass of cup #2	M_{2c}	$= (26.8 \pm 0.1) \times 10^{-3} \text{ kg}$
Distance cup #1 falls	h	$= (1.575 \pm 0.005) \text{ m}$
Total hanging mass	$M_1 + M_2 (= M_{1c} + M_{1a} + M_{2c} + M_{2a})$	$= (255.3 \pm 0.4) \times 10^{-3} \text{ kg}$

(Super Headers)

	Chosen Values	Observed Value	Calculated Values			
(Headers: Including description, symbol, units, and uncertainty if it is constant.)	Mass added to cup #1 M_{1a} (10^{-3} kg) $\pm 0.1\%$ (1)	Mass added to cup #2 M_{2a} (10^{-3} kg) $\pm 0.1\%$ (1)	Trial fall times t (s)	Average fall time (2) t_{ave} (s)	Acceleration a (m/s^2)	Difference in mass $M_1 - M_2$ (10^{-3} kg) ± 0.2
(Body: Including uncertainties if they vary.)	101	100	7.45 7.33 7.48 7.53	7.45 ± 0.10	0.057 ± 0.002	1.7
	102	99	4.77 4.68 4.71 4.81	4.74 ± 0.06	0.140 ± 0.004	3.7

	↓	↓	↓	↓	↓	↓

	107	95	2.54 2.65 2.69 2.59	2.62 ± 0.07	0.46 ± 0.03	11.7

(Notes)

- (1) Added masses were from a standard set of lab masses and were not individually measured.
- (2) No estimate was made for uncertainty in individual trials; uncertainty in average times were determined from variations among trials.

Appendix E: Graphs

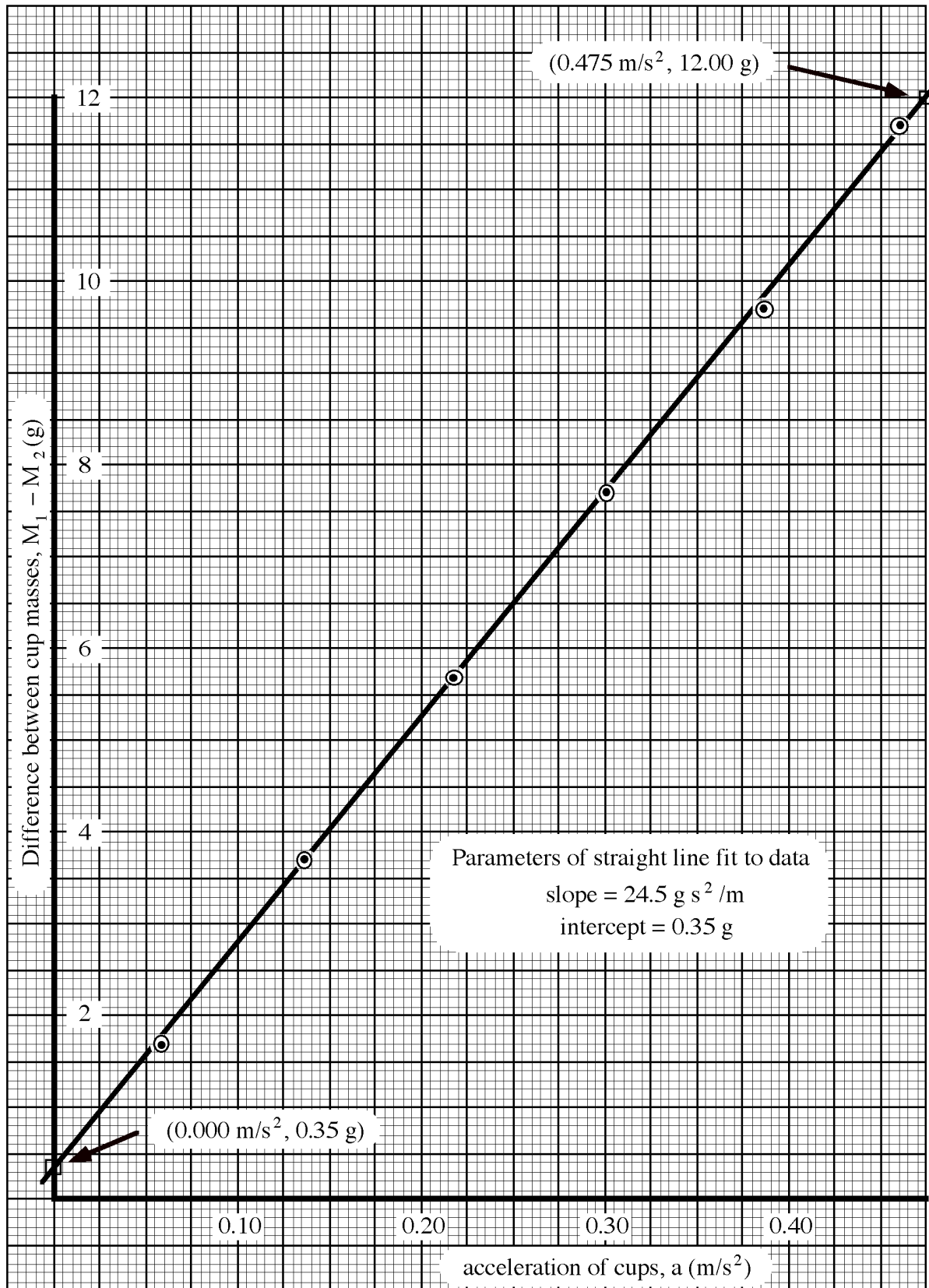
Graphs are used to discover and/or illustrate relationships between quantities. Please look at the example on the next page and note the following items:

- 1 Graphs have *meaningful* titles that include any important information that characterizes the data.
- 2 The scale for each axis is chosen so that the data will fill as much of the page as possible. This means that the minimum and maximum values on each axis should reflect the range of values to be plotted. It is *not* always necessary—and it is often *undesirable*—to include “zero” on one or both axes.
- 3 Subject to the previous considerations, scales should be chosen to *simplify* the placing of data points. Let major divisions of the graph paper represent increments of 1, 2, 4, or 5 in the least significant

digits of the scale values even if it means sacrificing a little of the graph area. Doing this will make it much easier to determine the values that correspond to the minor divisions. For example, consider plotting a value like 3.27 on the two scales shown at right. Because the minor divisions represent intervals of 0.14 on the first axis, it takes some thought to figure out where to place the data point. The job is *much* easier (and *much* less susceptible to errors!) on the second axis where the minor divisions represent intervals of 0.1.

- 4 Each axis is labeled with 1) a few words describing the thing being plotted, 2) its symbol (or *combination* of symbols), and 3) the units together with any factored out power of ten enclosed in parentheses.
- 5 Data points are enclosed in small circles (or boxes, etc.) to make them stand out and to clearly distinguish them from other marks.
- 6 If a straight line or some other kind of curve is fit to the data, it skips *over* the data points rather than being drawn right *through* them.
- 7 Straight line or curved “fits” to the data are never drawn by “connecting the dots.” Rather, they are drawn to indicate the apparent trend of the data, recognizing that experimental uncertainties will cause the data to “scatter” about the best fit.
- 8 Parameters of the fit are displayed directly on the graph *with units*. We will often fit straight lines to data and will, therefore, want to indicate the *slope* and *intercept* of the fit. The intercept can be read right off the graph (*unless* the horizontal axis does *not* include “zero”). To find the slope, determine the coordinates of two points on the line—*not* data points since they do not, in general, even *lie* on that line! Use them to determine the slope of the line. These points should be clearly indicated on the graph and should lie as far apart as possible in order to reduce the effects of round off error on the calculated slope.

Figure 2: Accelerating “force” (difference in cup masses) versus acceleration of cups with total mass held constant at 0.2553 kg.



“Least Squares” Linear Fits

How does one determine the “best” linear fit to a set of plotted data? It may often be sufficient to simply “eyeball” the data and get a line that “comes as close as possible to all of the data.” This statement may *seem* clear enough, but it certainly does *not* specify a rigorous procedure—one that would ensure that different people get the same answer. It is not hard to specify such a procedure. A very common one is called the “method of least squares.”

Suppose we consider a pair of possible values m and b for the slope and intercept (respectively) of a linear fit to our x, y data. We form the quantity S from the differences between our observed values of the y data (y_i) and those determined from the expression ($mx_i + b$) as follows:

$$S = \sum_{i=1}^N [y_i - (mx_i + b)]^2$$

That is, S is the *sum* of the *squares* of the “vertical” *differences* between our data and the straight line. (Squaring the differences eliminates any cancellation between differences that are positive and those that are negative.) Now we can define what we mean by the “best” fit: The best fit is characterized by the values of m and b that minimize S .

Determining the proper values of m and b requires calculus but the results can be written as follows:

The Method of Least Squares

To determine the slope and intercept of the “best” fit straight line

- 1 Calculate the four sums from the data values— x_i and y_i .

$$S_x = , \quad S_y = , \quad S_{xx} = , \quad S_{xy} = ,$$

- 2 Calculate the slope m and the intercept b .

$$m = \quad \text{and} \quad b =$$

where N is the number of data points.

Notice that when you use this method, there is no need to determine the coordinates of two points on the line as discussed in item 8 of the previous section. To be clear about how you obtain your values for the slope and intercept then, you should indicate that they were determined using the method of least squares.

Because this procedure is so simple and so common, most calculators have it “built in;” all the user needs to do is to enter the data values. If you don’t know whether your calculator does automatic least squares calculations, we *strongly* suggest that you read its manual. (The manual may refer to this procedure as “linear regression.”)

Uncertainties in Slopes and Intercepts

Now we know how to find the “best” values for the slope and the intercept, but we still need estimates for the *uncertainties* in these values. How might we get them? This turns out to be a *very* complicated subject in general; the answer depends on many assumptions about the nature of the data and the uncertainties in them. We will be satisfied for now with a method that is far from rigorous, but which gives *reasonable* answers and avoids the many complications.

Let us assume

1) that we have taken data with x -values ranging from x_{\min} to x_{\max} with

$$R \equiv x_{\max} - x_{\min} \text{ and}$$

$$x_{\text{mid}} \equiv (x_{\max} + x_{\min})/2,$$

2) that we have already found the slope m_{best} of the best fit straight line to the data, and

3) that the median uncertainty in our y -values is δy . (This may be determined from the previously estimated uncertainties in the data or from the actual deviations of the data points from the line.)

We expect that the “true” y -values lie within δy of the straight line. Therefore, the line with *maximum* slope that still reasonably fits our data runs from the point δy below the line at x_{\min} to the point δy above the line at x_{\max} . You can show that the slope of this line is $m_{\text{best}} + 2\delta y/R$ from which we infer that $\delta m \approx 2\delta y/R$. The intercept of this line is $b_{\text{best}} - 2\delta y x_{\text{mid}}/R$ from which we infer that $\delta b \approx x_{\text{mid}}\delta m$ (unless the data range *includes* $x=0$ in which case $\delta b \approx \delta y$). These results are summarized (and simplified) below.

Simple uncertainties in slopes and intercepts

Given x_{\min} and x_{\max} (the minimum and maximum x -values in the data) and δy (the median uncertainty in the data y -values we find

1 Uncertainty in slope: $\delta m \approx$

(This formula illustrates the importance of taking data over as *large* a range as possible.)

2 Uncertainty in intercept: $\delta b \approx \delta m$ or δy

(Use whichever one is larger)

Appendix F

Table I. Densities (grams/cm³)

	<u>Metals</u>		<u>Other Solids</u>
Aluminum	2.70	Glass, common	2.4 - 2.8
Brass	*8.4 -8.7	Graphite	1.9 - 2.3
Copper	8.94	Granite	2.64 - 2.76
Gold	19.3	Lucite	1.18 - 1.20
Iron	7.87	Marble	2.60 -2.84
Lead	11.34	Wood (seasoned)	
Silver	10.5	Birch	0.51 - 0.77
Steel	*7.8 -8.0	**Cork	0.22 - 0.26
Tin	7.30	Ebony	1.11 - 1.33
Zinc	7.14	Walnut	0.64 - 0.70

*Intervals account for difference in commercially available alloys

**Made of bark

Liquids

Alcohol, ethyl	0.791
Alcohol, methyl	0.810
Mercury	13.5
Turpentine	0.870
Water, pure	1.000
Water, sea	1.025

Sources of Densities: CRC Handbook of Chemistry and Physics, and O. W. Eschbach, Handbook of Engineering

Appendix G: Notes on Equipment

The Dial Caliper

Note that the scale on the shaft of the caliper is marked off in 10 mm divisions and the zero indicator can be used to determine a rough value for the length. One rotation of the dial pointer represents 10 mm and the smallest division on the dial is 0.1 mm. The caliper is “zeroed” by closing its jaws and rotating the outside of the dial until the needle points to “0.” The caliper may be used to measure outside diameters (with the larger pair of jaws), inside diameters (with the smaller pair of jaws), or depths (with the thick wire that extends out the end of the instrument.) The instrument may be held and the jaws moved with one hand using the thumb wheel. The jaws may be prevented from slipping using the lock. The reading of the caliper shown in Fig. 1 is about 22.2 mm.

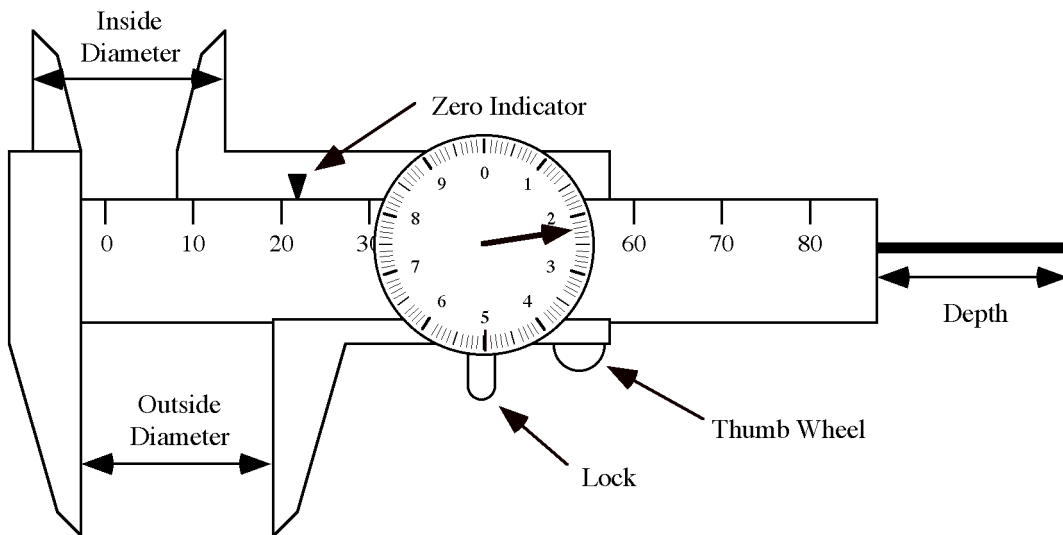


Fig. 1: The dial caliper.

The Micrometer

Note that the barrel is marked off in millimeters *and* half millimeters and that the smallest divisions on the thimble represent 0.01 mm. Because there are only 50 divisions around the circumference of the thimble, it takes *two* turns to move one millimeter along the barrel. Make sure that the micrometer is zeroed and, if it isn't, make a note of the reading at zero so that measured values can be corrected. Do not attempt to “zero” the micrometer yourself. Use the small frictional ratchet as you tighten the micrometer on the object being measured for consistency and to avoid overtightening. Read the barrel to the nearest 0.5 mm and add the thimble reading (which is indicated in 100th's of a mm). If necessary—i.e., if the instrument is not properly zeroed—subtract the zero reading. The reading of the micrometer shown in Fig. 2 is about 12.94 mm.

Fig. 2: The micrometer scale and ratchet.

