Numerical Methods Übungen

LU Decomposition, Spectral Norm, and Hessian Matrix

Problem 1

Consider the general 2×2 matrix:

$$[g_2] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1}$$

where a, b, c and d are real, $a > 0$, and $d > 0$.

a. First, factor the matrix $[g_2]$ above into the product of lower times an upper triangular matrix: $[g_2] = [l][u]$ using the Gauss-Doolittle method.

b. Next, factor the matrix $[g_2]$ above into the product of lower times an upper triangular matrix: $[g_2] = [l][u]$ using the Square Root method.

c. Check for both cases above that $|[l][u]|$ equals $ad - bc$.

Problem 2

Consider the Hermitian matrix below:

$$[a] = \begin{bmatrix} 1 & (1+i) & -i \\ (1-i) & 6 & (1+i) \\ i & (1-i) & 5 \end{bmatrix} \tag{2}$$

a. Factor $[a]$ into a lower times an upper triangular matrix using the Square Root method.

Problem 3

Consider the following matrix:

$$[a] = \begin{bmatrix} 1 & (1+i) & 0 \\ (1-i) & 4 & (2+i) \\ 0 & (2-i) & 1 \end{bmatrix} \tag{3}$$

a. What is the spectral norm of the matrix $[a]$?

Problem 4

Consider a system of two particles, each of mass m, that have the following potential energy function $V(x_1, x_2)$:

$$V(x_1, x_2) = \frac{c}{2}l^2(1 - cos(\frac{x_1 + 2x_2}{l}) + sin^2(\frac{x_1}{l}) + sin^2(\frac{x_2}{l})) \tag{4}$$

where x_1 and x_2 are the location of particle 1 and 2 on the x-axis. The constant c has units of force/length and l has units of length. The expansion point for our Taylor Series analysis will be $\vec{p} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

a. Determine V at the point $\vec{p}$.

b. Determine the gradient of the function V, $\vec{\nabla}V(x_1, x_2)$ at $\vec{p}$.

c. Determine the Hessian Matrix, $[h]_{ij} = \frac{\partial^2 V}{\partial x_i \partial x_j}$, at $\vec{p}$.

d. Find the eigenvalues of $[h]$.
Note: The equation of motion for the system for small oscillations about $\vec{p}$ is given by

$$[h] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = m\omega^2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5}$$

So, the eigenvalues of $[h]$ are related to the frequencies, ω, of the normal modes of oscillation. Try to determine the frequencies of the normal modes.