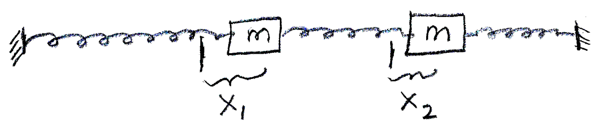
Numerical Methods Übungen
Eigenvalues and Eigenvectors

Problem 1
In the linear coupled spring system, one needs to solve the following "linear



eigenvalue" problem:

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad (1)$$

where $\lambda = -\omega^2 \frac{m}{c}$. Here ω is the angular frequency of the oscillation, c is the common spring constant, and m is the mass of each oscillating mass.

a. Solve for the eigenvalues λ and corresponding angular frequencies of the normal modes.

b. Solve for the eigenvectors. Describe the motion of the normal modes.

See the next page for a derivation of the eigenvalue equation.

The eigenvalue equation can be derived as follows. We apply Newton's second law to each object:

$$\begin{aligned} m\ddot{x}_1 &= -cx_1 + c(x_2 - x_1) \\ m\ddot{x}_2 &= -cx_2 - c(x_2 - x_1) \end{aligned}$$

After rearranging the terms we have

$$\begin{aligned} \ddot{x}_1 &= -2\frac{c}{m}x_1 + \frac{c}{m}x_2 \\ \ddot{x}_2 &= \frac{c}{m}x_1 - 2\frac{c}{m}x_2 \end{aligned}$$

We want a solution for which each mass has the same frequency f, i.e. angular frequency ω. Try:

$$x_1(t) = a_1\, cos(\omega t) \qquad x_2(t) = a_2\, cos(\omega t) \tag{2}$$

Substituting into the differential equations gives

$$\begin{aligned} -\omega^2 a_1 &= -2\frac{c}{m}a_1 + \frac{c}{m}a_2 \\ -\omega^2 a_2 &= \frac{c}{m}a_1 - 2\frac{c}{m}a_2 \end{aligned}$$

after canceling $cos(\omega t)$ common to each term. These equations can be rewritten in matrix form:

$$\frac{c}{m}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\omega^2\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{3}$$

or equivalently,

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\omega^2\frac{m}{c}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{4}$$

Problem 2

Consider the matrix $[a]$ below:

$$[a] = \begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix} \tag{5}$$

a. Find all the eigenvalues λ for the eigenvalue equations $[a]\vec{\phi} = \lambda\vec{\phi}$ and $\vec{\psi}^T[a] = \lambda\vec{\psi}^T$.

b. Determine the right eigenvectors $\vec{\phi}$ and the left eigenvectors $\vec{\psi}^T$ for each eigenvalue.

c. Check if the left and right eigenvectors for different eigenvalues are orthogonal: does $\vec{\psi}_i^T\vec{\phi}_j = 0$ if $i \neq j$.

d. Scale the left and right eigenvectors such that $\vec{\psi}_i^T\vec{\phi}_j = \delta_{ij}$

e. Check that the modal matrices $[\Psi]$ and $[\Phi]$ are inverses of each other.